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Loop Variables and Gauge Invariant Interactions - I *

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Abstract

We describe a method of writing down interacting equations for all the modes of the bosonic open string. It is a generalization of the loop variable approach that was used earlier for the free, and lowest order interacting cases. The generalization involves, as before, the introduction of a parameter to label the different strings involved in an interaction. The interacting string has thus becomes a "band" of finite width. The interaction equations expressed in terms of loop variables, has a simple invariance that is exact even off shell. A consistent definition of space-time fields requires the fields to be functions of all the infinite number of gauge coordinates (in addition to space time coordinates). The theory is formulated in one higher dimension, where the modes appear massless. The dimensional reduction that is needed to make contact with string theory (which has been discussed earlier for the free case) is not discussed here.

*This is a detailed description of an approach, outlined in a talk at the Puri Workshop in 1996, to use loop variables to string interactions.

1 Introduction

The loop variable approach introduced in [1] (hereafter I) (see also [3]) is an attempt to write down gauge invariant equations of motion for both massive and massless modes. This method being rooted in the sigma model approach [6, 7, 9, 8, 11, 10],the computations are expected to be simpler and the gauge transformation laws more transparent. This hope was borne out at the free level and also to a certain extent in the interacting case [2] (hereafter II). The gauge transformations at the free level can be summarized by the equation

$$k(t) \rightarrow k(t)\lambda(t) \tag{1.0.1}$$

Here $k(t)$ is the generalized momentum Fourier-conjugate to X and λ is the gauge parameter. This clearly has the form of a rescaling and one can speculate on the space-time interpretation of the string symmetries as has been done for instance in I.

In II the interacting case was discussed. It was shown that the leading interactions could be obtained by the simple trick of introducing an additional parameter 'σ' as $k(t) \rightarrow k(t,\sigma)$, parametrizing different interacting strings. Thus, for instance, $k_1^\mu(\sigma_1)k_1^\nu(\sigma_2)$ could stand for two massless photons when $\sigma_1 \neq \sigma_2$, but when $\sigma_1 = \sigma_2$ it would represent a massive "spin 2" excitation of one string. The gauge transformations admit a corresponding generalization

$$k(t,\sigma) \rightarrow k(t,\sigma) \int d\sigma_1 \lambda(t,\sigma_1) \tag{1.0.2}$$

It was shown, however that this prescription introduces only the leading interaction terms.[1]

In a talk some years ago [4] (hereafter III) we showed that there is a natural generalization of this construction to include the full set of interactions that one expects based on the operator product expansion (OPE) of vertex operators. It was shown that this construction gives gauge invariant equations. The generalization involves introducing σ-dependence in the X coordinates also. Gauge invariance at the level of loop variables is very easy to see. What was not clear at the time was whether there was a consistent map to space-time fields. Here we show that this is in fact the case. It crucially involves keeping a finite cutoff on the world sheet and also making

[1] Our notation, unfortunately, is perverse: The variable t originally used in the free theory lies *along* the string, and σ introduced in II labelling as it does the number of interaction vertices, parametrizes *evolution*.

the space-time fields a function of x_n. Keeping a finite cutoff is required when going off shell [19, 21, 22]. In the presence of a finite cutoff there are problems with gauge invariance as discussed in [13]. It was shown there that to lowest order these problems could be resolved by adding a massive mode with an appropriate transformation law. It was also speculated that maintaining exact gauge invariance would be possible if all the modes are kept. This is shown to be true in the present work. We have the full gauge invariance and it is not violated by a finite cutoff and the construction necessarily requires all the modes.

Another feature that emerges from the present work is that the space-time fields have to be functions of the gauge coordinates x_n. This is forced on us when we require that it be possible to define the gauge transformation laws for the space time fields in a consistent way. This does not introduce any new physical degrees since these can be gauge fixed. Nevertheless it is amusing to note that space-time has effectively become infinite dimensional.

In order to make precise contact with string theory one has to perform another step that we do not discuss in this paper. It involves generalizing to the interacting case the dimensional reduction that was done in I (for the free case). Given that the basic technique involves calculation of correlators of vertex operators on the world sheet we are more or less guaranteed that we will reproduce bosonic string amplitudes. What needs to be shown is that the dimensional reduction does not violate the gauge invariance. We reserve this issue for a future publication [15].

This paper is organized as follows. In section II we give a short review of [2] and elaborate on the role of the parameter σ. In section III we describe the generalization outlined in [4]. In section IV we discuss the gauge invariance of the Loop Variable. Section V contains some examples of equations of motion. Section VI discusses how one obtains the gauge transformation laws of the space-time fields. Section VII discusses the consistency issue and shows why the fields have to be functions of x_n. Section VIII contains a summary and some concluding remarks. An Appendix contains some details of a covariant Taylor expansion.

2 Review

2.1 Free theory

In I the following expression was the starting point to obtain the equations of motion at the free level:

$$e^A = e^{k_0^2\Sigma+\sum_{n>0} k_n.k_0 \frac{\partial}{\partial x_n}\Sigma+\sum_{n,m>0} k_n.k_m(\frac{\partial^2}{\partial x_n \partial x_m}-\frac{\partial}{\partial x_{n+m}})\Sigma+ik_nY_n} \tag{2.1.1}$$

The prescription was to vary w.r.t Σ and evaluate at $\Sigma = 0$ to get the equations of motion. Here, $2\Sigma \equiv < Y(z)Y(z) >$ and $Y = \sum_n \alpha_n \frac{\partial^n X}{(n-1)!} \equiv \sum_n \alpha_n \tilde{Y}_n$. α_n are the modes of the einbein $\alpha(t)$ used in defining the loop variable

$$e^{i\int_c \alpha(t)k(t)\partial_z X(z+t)dt+ik_0X} = e^{i\sum_n k_nY_n} \tag{2.1.2}$$

$$\alpha(t) = \sum_{n\geq 0} \alpha_n t^{-n}$$

$$k(t) = \sum_{n\geq 0} k_n t^{-n}$$

One can also show easily that $Y_n = \frac{\partial Y}{\partial x_n}$. Σ is thus a generalization of the Liouville mode, and what we have is a generalization of the Weyl invariance condition on vertex operators.

There is an alternative way to obtain the Σ dependence [12]. This is to perform a general conformal transformation on a vertex operator by acting on it with $e^{\sum_n \lambda_{-n}L_{+n}}$ using the relation [2][14]:

$$e^{\sum_n \lambda_{-n}L_{+n}} e^{iK_m\tilde{Y}_m} = e^{K_n.K_m\lambda_{-n-m}+\tilde{Y}_n\tilde{Y}_m\lambda_{+n+m}+imK_n\tilde{Y}_m\lambda_{-n+m}} e^{iK_m\tilde{Y}_m} \tag{2.1.3}$$

The anomalous term is $K_n.K_m\lambda_{-n-m}$ and the classical term is $mK_n\tilde{Y}_m\lambda_{-n+m}$. We will ignore the classical piece: this can be rewritten as a $(mass)^2$ term, which will be reproduced by performing a dimensional reduction, and other pieces involving derivatives of Σ (defined below) that correspond to field redefinitions [1]. We can apply (2.1.3) to the loop variable (5.3.31) by setting $K_m = \sum_n k_{m-n}\alpha_n$. Defining

$$\Sigma = \sum_{p,q} \alpha_p \alpha_q \lambda_{-p-q} \tag{2.1.4}$$

we recover (2.1.1). It is the approach described above that generalizes more easily to the interacting case.

[2]This relation is only true to lowest order in λ. The exact expression is given in [14]

The equations thus obtained are invariant under

$$k_n \to \sum_m k_{n-m}\lambda_m \tag{2.1.5}$$

which is just the mode expansion of (1.0.1).

That this is an invariance of the equations of motion derived from (2.1.1) follows essentially from the fact that the transformation (2.1.5), applied to (2.1.1) changes it by a total derivative.

$$\delta A = \sum_n \lambda_n \frac{\partial}{\partial x_n}[A] \tag{2.1.6}$$

The equations are obtained by the operation $\frac{\delta}{\delta\Sigma}A \mid_{\Sigma=0}$. Thus consider the gauge variation of this:

$$\delta_{gauge}\frac{\delta}{\delta\Sigma}A = \frac{\delta}{\delta\Sigma}\delta_{gauge}A$$

$$= \frac{\delta}{\delta\Sigma}\lambda_n\frac{\partial}{\partial x_n}A$$

Now A being linear in Σ and its derivatives can always be expressed after integration by parts as ΣB for some B. Thus we have

$$= \frac{\delta}{\delta\Sigma}\lambda_n\frac{\partial}{\partial x_n}(\Sigma B) = \lambda_n(-\frac{\partial}{\partial x_n}B + \frac{\partial}{\partial x_n}B) = 0$$

Thus the equations obtained from (2.1.1) are invariant.

The connection between these variables and transformation laws and the usual fields and gauge transformations was described in I. Briefly, the fields were defined by

$$S^{\mu\nu..}_{n,m,...}(k_0) =< k_n^\mu k_m^\nu ... >= \int [\prod_n dk_n d\lambda_n] k_n^\mu k_m^\nu ... \Psi[k_0,k_1,k_2,...,k_n,...\lambda_m...] \tag{2.1.7}$$

where Ψ is some "string field" that describes a given configuration.

And the gauge parameters $\Lambda^{\mu,\nu}_{p,n,m..}(k_0)$ were defined by a similar equation involving one power of λ_p, $p = 1,2,...$, and arbitrary numbers of $k_n, k_m..$ in the integrand.

However there are some caveats. In proving (2.1.6) one needs to use equations such as

$$\frac{\partial}{\partial x_1}(\frac{\partial^2}{\partial x_1^2} - \frac{\partial}{\partial x_2})\Sigma = (\frac{\partial^3}{\partial x_1^3} - \frac{\partial^2}{\partial x_1 \partial x_2})\Sigma = 2(\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial}{\partial x_3})\Sigma \tag{2.1.8}$$

which follow from the basic definitions [1]. This implies that equations of motion obtained by varying Σ will not be invariant. To see this consider the following expression:

$$2(\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial}{\partial x_3})\Sigma A + (\frac{\partial^2}{\partial x_1^2} - \frac{\partial}{\partial x_2})\Sigma \frac{\partial A}{\partial x_1} \tag{2.1.9}$$

Using (2.1.8) we get

$$= \frac{\partial}{\partial x_1}[(\frac{\partial^2}{\partial x_1^2} - \frac{\partial}{\partial x_2})\Sigma A] \tag{2.1.10}$$

which is a total derivative. However if we vary (2.1.9) w.r.t. Σ, one gets

$$2\delta\Sigma(\frac{\partial^2}{\partial x_1 \partial x_2} + \frac{\partial}{\partial x_3})A + \delta\Sigma(\frac{\partial^2}{\partial x_1^2} + \frac{\partial}{\partial x_2})\frac{\partial A}{\partial x_1} \tag{2.1.11}$$

which is not zero. On the other hand if we rewrite (2.1.9) as (using (2.1.8))

$$(\frac{\partial^3}{\partial x_1^3} - \frac{\partial^2}{\partial x_1 \partial x_2})\Sigma A + (\frac{\partial^2}{\partial x_1^2} - \frac{\partial}{\partial x_2})\Sigma \frac{\partial A}{\partial x_1} \tag{2.1.12}$$

and vary w.r.t Σ we get

$$\delta\Sigma(-\frac{\partial^3}{\partial x_1^3} - \frac{\partial^2}{\partial x_1 \partial x_2})A + \delta\Sigma(\frac{\partial^3}{\partial x_1^3} + \frac{\partial^2}{\partial x_1 \partial x_2})A \tag{2.1.13}$$

which is zero.

Thus one has to be careful about varying w.r.t Σ indiscriminately. Let us review the solution to this as we will face the same issue in the interacting case discussed in the next section. Consider the variation of the exponent A (2.1.1), reproduced below, due to λ_p:

$$e^A = e^{k_0^2\Sigma + \sum_{n>0} k_n.k_0 \frac{\partial}{\partial x_n}\Sigma + \sum_{n,m>0} k_n.k_m(\frac{\partial^2}{\partial x_n \partial x_m} - \frac{\partial}{\partial x_{n+m}})\Sigma + ik_nY_n} \tag{2.1.14}$$

The change is

$$\lambda_p(\sum_n k_{n-p}.k_0 \frac{\partial}{\partial x_n}\Sigma + \sum_{n,m\neq p} k_{n-p}.k_m(\frac{\partial^2}{\partial x_n \partial x_m} - \frac{\partial}{\partial x_{n+m}})\Sigma +$$

$$+ \sum_m k_m.k_0(\frac{\partial^2}{\partial x_m \partial x_p} - \frac{\partial}{\partial x_{m+p}})\Sigma + i\sum_n k_{n-p}Y_n) \tag{2.1.15}$$

If we assume tracelessness of the gauge parameter so that any term of the form $\lambda_p k_n.k_m$ is zero then the second sum in (2.1.15) vanishes and using the fact that the first sum cancels the second term in the last sum we can rewrite the variation of A as

$$\lambda_p \frac{\partial}{\partial x_p}\{\sum_m k_m.k_0 \frac{\partial}{\partial x_m}\Sigma + \sum_n ik_{n-p}Y_{n-p} + \sum_{n,m} k_n.k_m(\frac{\partial^2}{\partial x_n \partial x_m} - \frac{\partial}{\partial x_{n+m}})\Sigma\}$$
$$= \lambda_p \frac{\partial}{\partial x_p} A \tag{2.1.16}$$

Note that in the first line of this equation we have added a term that vanishes by the tracelessness constraint, *viz* terms involving $\lambda_p k_n.k_m$. But it is important that we have *not* used identities of the type given in (2.1.8). Thus tracelessness of the gauge parameters ensures the gauge invariance of the equations.

2.2 Interactions

In II this approach was generalized to include some interactions. The basic idea was to introduce a new parameter $\sigma : 0 \le \sigma \le 1$ to label different strings and to replace each k_n in the free equation by $\int_0^1 d\sigma k_n(\sigma)$. The next step was to assume that

$$< k_1^\mu(\sigma_1) k_1^\nu(\sigma_2) >= S^{\mu\nu}\delta(\sigma_1 - \sigma_2) + A^\mu A^\nu \tag{2.2.17}$$

where $< ... >$ denotes $\int \mathcal{D}k(\sigma)...\Psi[k(\sigma)]$, Ψ being the"string field" defined in I.[3] This corresponds to saying that when $\sigma_1 = \sigma_2$, both the k_1's belong to the same string and otherwise to different strings where they represent two photons at an interaction point.[4] The gauge transformation is replaced by (1.0.2). This is easily seen to give interacting interacting equations. However the fact is that this is only a leading term in the infinite set of interaction vertices.

As a prelude to generalizing this construction, let us explain more precisely the nature of the replacement $k_n \to \int_0^1 d\sigma k_n(\sigma)$. Let us split the interval $(0,1)$ into N bits of width $a = \frac{1}{N}$. We will assume that when σ

[3] No special property of Ψ is assumed other than this.

[4] It will be seen that (2.2.17)has to be generalized by replacing the δ-function on the RHS by something else, when we attempt to reproduce string amplitudes [19]. However in this paper we will not do so.

satisfies $\frac{n}{N} \leq \sigma \leq \frac{n+1}{N}$ it represents the $(n+1)$th string. Let us also define a function

$$\begin{aligned} D(\sigma_1, \sigma_2) &= 1\, if\ \sigma_1, \sigma_2\ belong\ to\ the\ same\ interval \\ &= 0\, if\ \sigma_1\ , \sigma_2\ belong\ to\ different\ intervals. \end{aligned} \tag{2.2.18}$$

Thus $\int_0^1 d\sigma_1 D(\sigma_1 \sigma_2) = \ a \ = \int_0^1 d\sigma_1 \int_0^1 d\sigma_2 D(\sigma_1 \sigma_2)$.

Then we set

$$< k^{\mu}(\sigma_1) k^{\nu}(\sigma_2) >= \frac{D(\sigma_1, \sigma_2)}{a} S^{\mu\nu} + A^{\mu} A^{\nu} \tag{2.2.19}$$

In the limit $N \to \infty,\ a \to 0,\ \frac{D(\sigma_1,\sigma_2)}{a} \approx \delta(\sigma_1 - \sigma_2)$ and we recover (2.2.17).

In effect (2.1.1) has been modified to

$$e^{\int_0^1 \int_0^1 d\sigma_1 d\sigma_2 [k_0(\sigma_1) k_0(\sigma_2) \Sigma + k_n(\sigma_1).k_0(\sigma_2) \frac{\partial}{\partial x_n} \Sigma + \sum_{n,m} (\frac{\partial^2}{\partial x_n \partial x_m} - \frac{\partial}{\partial x_{n+m}}) \Sigma] + \int_0^1 d\sigma k_n(\sigma) Y_n} \tag{2.2.20}$$

The final step (which is also necessary in the free case), is to dimensionally reduce to obtain the massive equations. For details we refer the reader to I.

The modification (2.2.17), that replaces $S^{\mu\nu}$ by $S^{\mu\nu} + A^{\mu}A^{\nu}$ can be understood in terms of the OPE. Consider a correlation function involving two vector vertex operators and any other set of operators, that we represent as

$$\mathcal{A} =< V_1 V_2 ... V_N : k_1^{\mu} \partial_z X^{\mu} e^{i \int k_0 Y} : q_1^{\nu} \partial_w X^{\nu} e^{i q_0 Y} > \tag{2.2.21}$$

The OPE of $: k_1^{\mu} \partial_z X^{\mu}(z) e^{i \int k_0 Y} :$ and $: q_1^{\nu} \partial_w X^{\nu}(w) e^{i q_0 Y} :$

is given by

$$\begin{gathered} : k_1^{\mu} \partial_z X^{\mu}(z) e^{i \int k_0 Y} :: q_1^{\nu} \partial_w X^{\nu}(w) e^{i q_0 Y} := \\ : k_1^{\mu} q_1^{\nu} \partial_z X^{\mu} \partial_w X^{\nu} e^{i(k_0 X(z) + q_0 X(w))} : + \, terms\ involving\ contractions. \end{gathered} \tag{2.2.22}$$

We can Taylor expand

$$X(w) = X(z) + (w - z) \partial_z X + O(w - z) + ... \tag{2.2.23}$$

This gives for the leading term in (2.2.21)

$$\mathcal{A} =< V_1 V_2 ... V_N : k_1^{\mu} q_1^{\nu} \partial_z X^{\mu} \partial_z X^{\nu} e^{i(k_0 X(z) + q_0 X(w))} :> \tag{2.2.24}$$

Compare this with the correlation involving $S^{\mu\nu}$:

$$\mathcal{A}' =< V_1 V_2 ... V_N : k_1^{\mu} k_1^{\nu} \partial_z X^{\mu} \partial_z X^{\nu} e^{i \int k_0 Y} :> \tag{2.2.25}$$

We see that $\mathcal{A}$ and $\mathcal{A}'$ give identical terms except that $S^{\mu\nu}$ is replaced by $A^\mu A^\nu$. It is in this sense that the substitution given in II, gives the leading term in the OPE. The crucial point is that, while in (2.2.20) we have introduced the parameter σ in the k_n's we have not done so for the Y_n's. This is equivalent to approximating $X(w)$ by $X(z)$ in (2.2.23). Clearly, the generalization required to get all the terms is to introduce the parameter σ in Y also. We turn to this in the next section.

3 Interactions

3.1 Introducing σ-dependence in the loop variable

We will introduce the parameter σ in all the variables keeping in mind the basic motivation that σ labels different vertex operators. Thus all the variables that are required to define a vertex operator become σ dependent. Thus

$$X^\mu(z) \rightarrow X^\mu(z(\sigma)) \tag{3.1.1}$$

$$x_n \rightarrow x_n(\sigma) \tag{3.1.2}$$

in addition to

$$k_n^\mu \rightarrow k_n^\mu(\sigma) \tag{3.1.3}$$

The σ-dependence of x_n in eqn. 3.1.2 is only an intermediate step. At the end of the day (but before any integration by parts is done) we will set all the x_n's to be the same. One can think of this merely as a device for keeping track of which term is being differentiated.

(3.1.1) and (3.1.2) imply that

$$\frac{\partial}{\partial x_n} Y \rightarrow \frac{\partial}{\partial x_n(\sigma)} Y(z(\sigma), x_n(\sigma)) \tag{3.1.4}$$

Note that X need not be an explicit function of σ since at a given location z, on the world sheet there can only be one $X(z)$. As an example of the above consider the case when we have regions $(0, 1/2)$ and $(1/2, 1)$. When $0 \leq \sigma \leq 1/2$ one has $z(\sigma) \equiv z$ and for $1/2 \leq \sigma \leq 1$ one has $z(\sigma) \equiv w$. Similarly $x_n(\sigma)$ could be called x_n, y_n in the two regions and $k_n(\sigma)$ could be called k_n, p_n in the two regions. Thus in this example the vertex operator $k_n(\sigma)Y_n(z(\sigma), x_n(\sigma))e^{ik_0(\sigma)Y(\sigma)}$ stands for $k_n \frac{\partial Y}{\partial x_n}(z, x_i)e^{ik_0 Y(z, x_n)}$ and $p_n \frac{\partial Y}{\partial y_n}(w, y_i)e^{ip_0 Y(w, y_n)}$ in the two regions.

Now we have to clarify what we mean by a derivative w.r.t $x_n(\sigma)$: In (3.1.4) we have $\frac{\partial Y(z(\sigma),x_i(\sigma))}{\partial x_n(\sigma)}$: One has to specify the meaning of $\frac{\partial x_n(\sigma)}{\partial x_n(\sigma')}$. Clearly what we want is: If σ, σ' belong to the same interval, then $\frac{\partial x_n(\sigma)}{\partial x_n(\sigma')} =$ 1 and zero otherwise. Thus using (2.2.18)

$$\frac{\partial x_n(\sigma)}{\partial x_n(\sigma')} = D(\sigma, \sigma') \tag{3.1.5}$$

or more generally

$$\frac{\partial x_n(\sigma)}{\partial x_m(\sigma')} = \delta_{nm} D(\sigma, \sigma') \tag{3.1.6}$$

Note that this is not the same as the conventional functional derivative. However we can define

$$\frac{\delta x_n(\sigma)}{\delta x_n(\sigma')} \equiv \frac{D(\sigma, \sigma')}{a} \tag{3.1.7}$$

which, in the limit $a \rightarrow 0$ becomes the usual functional derivative. Thus

$$\int d\sigma' \frac{\delta Y(\sigma)}{\delta x_n(\sigma')} = \frac{\partial Y(\sigma)}{\partial x_n(\sigma)} \tag{3.1.8}$$

We can now write down the generalization of (2.1.1)

$$\begin{gathered} exp\{\int\int d\sigma_1 d\sigma_2 \{k_0(\sigma_1).k_0(\sigma_2)[\tilde{\Sigma}(\sigma_1,\sigma_2) + \tilde{G}(\sigma_1,\sigma_2)] \\ + \int\int d\sigma_3 d\sigma_4 \sum_{n,m\geq 0} k_n(\sigma_1).k_m(\sigma_2) \\ \frac{1}{2}[\frac{\delta^2}{\delta x_n(\sigma_1)\delta x_m(\sigma_2)} - \delta(\sigma_1 - \sigma_2)\frac{\delta}{\delta x_{n+m}(\sigma_1)}][\tilde{\Sigma}(\sigma_3,\sigma_4) + \tilde{G}(\sigma_3,\sigma_4)]\}\} \\ exp\{i\int d\sigma k_n(\sigma) Y_n(\sigma)\} \end{gathered} \tag{3.1.9}$$

For convenience of notation have assumed the following:

$$\frac{\delta \alpha_n(\sigma_1)}{\delta x_0(\sigma_2)} = \frac{D(\sigma_1 - \sigma_2)}{a}\alpha_n(\sigma_1)$$

This saves us the trouble of writing separately the case $n = 0$ in the sum in (3.1.9).

In (3.1.9) $G(\sigma_1, \sigma_2) = \tilde{G}(z(\sigma_1), z(\sigma_2)) =< Y(z(\sigma_1))Y(z(\sigma_2)) >$ is the Green function which starts out as $ln(z_1 - z_2)$. We have suppressed the

Lorentz indices. One might expect by Lorentz invariance $\tilde{G}^{\mu\nu} = \delta^{\mu\nu}\tilde{G}$. However in I it was seen that the $D+1$th coordinate has a special role and is like the ghost coordinate of bosonic string theory. So there is no reason to expect the full $SO(D+1)$ invariance. In fact [15] we will have to assume some specific properties for $\tilde{G}^{D+1,D+1}$ in order to reproduce string amplitudes.

More precisely, if we define: [Using the notation $z_i = z(\sigma_i)$]

$$D_{z_1} = D_{z(\sigma_1)} \equiv 1 + \alpha_1(\sigma_1)\frac{\partial}{\partial z(\sigma_1)} + \alpha_2\frac{\partial^2}{\partial z^2(\sigma_1)} + ... \tag{3.1.10}$$

so that

$$Y(z(\sigma)) = D_{z(\sigma)}X(z(\sigma)) \tag{3.1.11}$$

then,

$$\tilde{G}(z_1, z_2) = D_{z_1}D_{z_2}G(z_1, z_2) \tag{3.1.12}$$

$$\tilde{\Sigma}(\sigma_1, \sigma_2) = D_{z_1}D_{z_2}\rho(\sigma_1, \sigma_2) \tag{3.1.13}$$

where

$$\rho(\sigma_1, \sigma_2) = \frac{\lambda(z(\sigma_1)) - \lambda(z(\sigma_2))}{z(\sigma_1) - z(\sigma_2)} \tag{3.1.14}$$

is the generalization of the usual Liouville mode $\rho(\sigma)$ which is equal to $\frac{d\lambda}{dz}$. The $\tilde{\Sigma}$ dependence in (3.1.9) is obtained by the following step:

$$e^{:\frac{1}{2}\int du\lambda(u)[\partial_z X(z+u)]^2:}e^{ik_n\frac{\partial}{\partial x_n}D_{z_1}X}e^{ip_m\frac{\partial}{\partial x_m}D_{z_2}X} \tag{3.1.15}$$

defines the action of the Virasoro generators on the two sets of vertex operators.

$$= e^{ik_n\cdot p_m\partial_{x_n}\partial_{y_m}D_{z_1}D_{z_2}\oint du\frac{\lambda(u)}{z_1-z_2}[\frac{1}{z_1-u}-\frac{1}{z_2-u}]} \tag{3.1.16}$$

$$= e^{ik_n\cdot p_m\partial_{x_n}\partial_{y_m}\tilde{\Sigma}} \tag{3.1.17}$$

This expression is only valid to lowest order in λ which is all we need here.[5]. The expression

$$\int\int d\sigma_1 d\sigma_2\frac{1}{2}[\frac{\delta^2}{\delta x_n(\sigma_1)\delta x_m(\sigma_2)} - \delta(\sigma_1-\sigma_2)\frac{\delta}{\delta x_{n+m}(\sigma_1)}][\tilde{\Sigma}(\sigma_3,\sigma_4)+\tilde{G}(\sigma_3,\sigma_4)] \tag{3.1.18}$$

[5] The exact expression is given in [14]

can easily be seen to be equal to

$$\frac{\partial^2}{\partial x_n(\sigma_3)\partial x_m(\sigma_3)}\tilde{\Sigma}(\sigma_3,\sigma_4) \tag{3.1.19}$$

In the limit $\sigma_3=\sigma_4=\sigma$ this is just equal to $1/2[\frac{\partial^2}{\partial x_n(\sigma)\partial x_m(\sigma)} - \frac{\partial}{\partial x_{m+n}(\sigma)}]\tilde{\Sigma}(\sigma,\sigma)$ and reduces to the free field case described by (2.2.20)(provided the limit is taken after differentiation).

Let us show that the gauge transformation (1.0.2) changes (3.1.9) by a total derivative

$$\delta A = \int d\sigma'\lambda(\sigma')\int d\sigma \frac{\delta}{\delta x_n(\sigma)}A \tag{3.1.20}$$

4 Invariance of the Loop Variable

Our starting point is the loop variable e^A given by:

$$e^{i\{\int d\sigma k_0(\sigma)Y(\sigma)+i\sum_{n>0}k_n(\sigma)\frac{\partial Y(\sigma)}{x_n(\sigma)}\}}$$
$$e^{\int\int d\sigma_1 d\sigma_2\{k_0(\sigma_1)k_0(\sigma_2)[\tilde{\Sigma}+\tilde{G}](\sigma_1,\sigma_2)+(\sum_{n>0}k_n(\sigma_1).k_0(\sigma_2)\frac{\partial[\tilde{\Sigma}+\tilde{G}](\sigma_1,\sigma_2)}{\partial x_n(\sigma_1)}+\sigma_1\leftrightarrow\sigma_2)\}}$$
$$e^{\int\int d\sigma_1 d\sigma_2\{\sum_{n,m>0}k_n(\sigma_1).k_m(\sigma_2)\frac{\partial^2[\tilde{\Sigma}+\tilde{G}](\sigma_1,\sigma_2)}{\partial x_1(\sigma_1)\partial x_1(\sigma_2)}\}} \tag{4.0.1}$$

Under a gauge transformation:

$$k_n(\sigma_1)\rightarrow \int d\sigma\lambda_p(\sigma)k_{n-p}(\sigma_1) \tag{4.0.2}$$

Let us consider $p=1$.

$$k_1(\sigma_1).k_0(\sigma_2)\frac{\partial}{\partial x_1(\sigma_1)}[\tilde{\Sigma}+\tilde{G}](\sigma_1,\sigma_2)\rightarrow \int d\sigma\lambda_1(\sigma)k_0(\sigma_1).k_0(\sigma_2)\frac{\partial}{\partial x_1(\sigma_1)}[\tilde{\Sigma}+\tilde{G}](\sigma_1,\sigma_2) \tag{4.0.3}$$

$$k_0(\sigma_1).k_1(\sigma_2)\frac{\partial}{\partial x_1(\sigma_1)}[\tilde{\Sigma}+\tilde{G}](\sigma_1,\sigma_2)\rightarrow \int d\sigma\lambda_1(\sigma)k_0(\sigma_1).k_0(\sigma_2)\frac{\partial}{\partial x_1(\sigma_2)}[\tilde{\Sigma}+\tilde{G}](\sigma_1,\sigma_2) \tag{4.0.4}$$

Adding the two we get:

$$\int d\sigma\lambda_1(\sigma)[\frac{\partial}{\partial x_1(\sigma_1)}+\frac{\partial}{\partial x_1(\sigma_2)}]k_0(\sigma_1).k_0(\sigma_2)[\tilde{\Sigma}+\tilde{G}](\sigma_1,\sigma_2) \tag{4.0.5}$$

Similarly,

$$k_2(\sigma_1).k_0(\sigma_2)\frac{\partial}{\partial x_2(\sigma_1)}[\tilde{\Sigma}+\tilde{G}](\sigma_1,\sigma_2) \rightarrow \int d\sigma\lambda_1(\sigma)k_1(\sigma_1).k_0(\sigma_2)\frac{\partial}{\partial x_2(\sigma_1)}[\tilde{\Sigma}+\tilde{G}](\sigma_1,\sigma_2) \tag{4.0.6}$$

$$k_0(\sigma_1).k_2(\sigma_2)\frac{\partial}{\partial x_2(\sigma_2)}[\tilde{\Sigma}+\tilde{G}](\sigma_1,\sigma_2) \rightarrow \int d\sigma\lambda_1(\sigma)k_0(\sigma_1).k_1(\sigma_2)\frac{\partial}{\partial x_2(\sigma_2)}[\tilde{\Sigma}+\tilde{G}](\sigma_1,\sigma_2) \tag{4.0.7}$$

$$k_1(\sigma_1).k_1(\sigma_2)\frac{\partial^2[\tilde{\Sigma}+\tilde{G}](\sigma_1,\sigma_2)}{\partial x_1(\sigma_1)\partial x_1(\sigma_2)} \rightarrow \int d\sigma\lambda_1(\sigma)(k_1(\sigma_1).k_0(\sigma_2)+k_0(\sigma_1).k_1(\sigma_2))\frac{\partial^2[\tilde{\Sigma}+\tilde{G}](\sigma_1,\sigma_2)}{\partial x_1(\sigma_1)\partial x_1(\sigma_2)} \tag{4.0.8}$$

Adding we get,

$$\int d\sigma\lambda_1(\sigma)\{[\frac{\partial}{\partial x_1(\sigma_1)}+\frac{\partial}{\partial x_1(\sigma_2)}]k_1(\sigma_1).k_0(\sigma_2)\frac{\partial}{\partial x_1(\sigma_1)}[\tilde{\Sigma}+\tilde{G}]+$$
$$[\frac{\partial}{\partial x_1(\sigma_1)}+\frac{\partial}{\partial x_1(\sigma_2)}]k_0(\sigma_1).k_1(\sigma_2)\frac{\partial}{\partial x_1(\sigma_2)}[\tilde{\Sigma}+\tilde{G}]\} \tag{4.0.9}$$

$$= \int d\sigma\lambda_1(\sigma)[\frac{\partial}{\partial x_1(\sigma_1)}+\frac{\partial}{\partial x_1(\sigma_2)}]\{k_1(\sigma_1).k_0(\sigma_2)\frac{\partial}{\partial x_1(\sigma_1)}[\tilde{\Sigma}+\tilde{G}]+\sigma_1\leftrightarrow\sigma_2\} \tag{4.0.10}$$

Now consider

$$k_2(\sigma_1).k_1(\sigma_2)\frac{\partial^2[\tilde{\Sigma}+\tilde{G}]}{\partial x_2(\sigma_1)\partial x_1(\sigma_2)}+k_1(\sigma_1).k_2(\sigma_2)\frac{\partial^2[\tilde{\Sigma}+\tilde{G}]}{\partial x_1(\sigma_1)\partial x_2(\sigma_2)} \tag{4.0.11}$$

$$\rightarrow \int d\sigma\lambda_1(\sigma)k_1(\sigma_1).k_1(\sigma_2)\frac{\partial^2[\tilde{\Sigma}+\tilde{G}]}{\partial x_2(\sigma_1)\partial x_1(\sigma_2)}+\int d\sigma\lambda_1(\sigma)k_1(\sigma_1).k_1(\sigma_2)\frac{\partial^2[\tilde{\Sigma}+\tilde{G}]}{\partial x_1(\sigma_1)\partial x_2(\sigma_2)} \tag{4.0.12}$$

$$= \int d\sigma\lambda_1(\sigma)[\frac{\partial}{\partial x_1(\sigma_1)}+\frac{\partial}{\partial x_1(\sigma_2)}]k_1(\sigma_1).k_1(\sigma_2)\frac{\partial^2[\tilde{\Sigma}+\tilde{G}](\sigma_1,\sigma_2)}{\partial x_1(\sigma_1)\partial x_1(\sigma_2)} \tag{4.0.13}$$

From the above it is clear that we get the following:

$$\delta A = \int d\sigma \lambda_1(\sigma)[\frac{\partial}{\partial x_1(\sigma_1)} + \frac{\partial}{\partial x_1(\sigma_2)}]A \tag{4.0.14}$$

On setting $x_n(\sigma_1) = x_n(\sigma_2) = x_n$ we get

$$\delta A = \lambda_1(\sigma)\frac{\partial}{\partial x_1}A \tag{4.0.15}$$

Thus to lowest order in λ_1, e^A changes by a total derivative in x_1. This is obviously true for λ_p also.

Thus the equations obtained by varying w.r.t $\tilde{\Sigma}(z(\sigma_1), z(\sigma_2), x_n(\sigma_1), x_n(\sigma_2))$ are invariant. However $\tilde{\Sigma}$ is not a local field on the world sheet. A has terms of the form $[\frac{\partial^2}{\partial x \partial y}\Sigma(w,z,x,y)] \mid_{x=y} \neq \frac{\partial^2}{\partial x^2}[\Sigma(w,z,x,y) \mid_{x=y}]$. Thus A cannot be expressed in terms of x_n-derivatives of a field. We would have to use both x_n and Y_n. But we cannot integrate by parts on both x_n, y_n - there is no such gauge invariance. So we first Taylor expand it in powers of $z(\sigma_2) - z(\sigma_1)$ the coefficients of which are derivatives of a local field $\Sigma(z,x) \equiv \bar{\Sigma}(z,x,y) \mid_{x=y}$, where $\bar{\Sigma}(v,x,y) = \tilde{\Sigma}(v,v,x,y)$. Below we have used the letter v to denote $\frac{z(\sigma_1)+z(\sigma_2)}{2}$ and $x(\sigma_1) = x, x(\sigma_2) = y$.

$$\tilde{\Sigma}(\sigma_1,\sigma_2) = \bar{\Sigma}(v) + aD_1(x,y)\bar{\Sigma}(v) + a^2 D_2(x,y)\bar{\Sigma}(v) + ... \tag{4.0.16}$$

$$= \bar{\Sigma}(v) + a\sum_r(\gamma_r'^0\frac{\partial\bar{\Sigma}}{\partial y_{r+1}} - \gamma_r^0\frac{\partial\bar{\Sigma}}{\partial x_{r+1}}) + \tag{4.0.17}$$

$$\frac{a^2}{2!}[\sum_s(\gamma_s'^1\frac{\partial\bar{\Sigma}}{\partial y_{s+1}} + \gamma_s^1\frac{\partial\bar{\Sigma}}{\partial x_{s+1}}) - 2\sum_{r,s}\gamma_r^0\gamma_s'^0\frac{\partial^2\bar{\Sigma}}{\partial x_{r+1}\partial y_{s+1}}] + ...$$

D_k and γ are defined in the Appendix. Very explicitly, the first few terms of the Taylor expansion are :

$$\tilde{\Sigma}(\sigma_1,\sigma_2) = \bar{\Sigma}(v) + a\underbrace{[\frac{\partial\bar{\Sigma}}{\partial y_1} - \frac{\partial\bar{\Sigma}}{\partial x_1} + (\frac{y_1^2}{2} + y_2)\frac{\partial\bar{\Sigma}}{\partial y_3} - (\frac{x_1^2}{2} + x_2)\frac{\partial\bar{\Sigma}}{\partial x_3}]}_{D_1(x,y)} +$$

$$\frac{a^2}{2}\underbrace{[\frac{\partial\bar{\Sigma}}{\partial x_2} + \frac{\partial\bar{\Sigma}}{\partial y_2} - 2\frac{\partial^2\bar{\Sigma}}{\partial x_1\partial y_1} + x_1\frac{\partial\bar{\Sigma}}{\partial x_3} + y_1\frac{\partial\bar{\Sigma}}{\partial y_3} - 2(\frac{x_1^2}{2} + x_2)\frac{\partial^2\bar{\Sigma}}{\partial x_3\partial y_1} - 2(\frac{y_1^2}{2} + y_2)\frac{\partial^2\bar{\Sigma}}{\partial y_3\partial x_1}]}_{D_2(x,y)} + ... \tag{4.0.18}$$

Once you Taylor expand Σ we have the following problem that we encounter also in the free case. The problem is that when the gauge variation does not produce k_0 we need constraints:

$$k_2.k_1[\frac{\partial^2\Sigma}{\partial x_2\partial x_1} - \frac{\partial\Sigma}{\partial x_3}] \to \lambda_1 k_1.k_1[\frac{\partial^2\Sigma}{\partial x_2\partial x_1} - \frac{\partial\Sigma}{\partial x_3}]$$

$$?=?\lambda_1 k_1.k_1\frac{\partial}{\partial x_1}[\frac{\partial^2\Sigma}{\partial x_1^2} - \frac{\partial\Sigma}{\partial x_2}] \tag{4.0.19}$$

The last equation is not an identity and follows only because certain constraints are obeyed by Σ. This in turn requires the imposition of the constraints on the gauge parameters - $\lambda_1 k_1.k_1 = 0$.

This problem is not there for the λ_2 variation as can be seen in the following:

$$k_2.k_1[\frac{\partial^2\Sigma}{\partial x_2\partial x_1} - \frac{\partial\Sigma}{\partial x_3}] \to \lambda_2 k_0.k_1[\frac{\partial^2\Sigma}{\partial x_2\partial x_1} - \frac{\partial\Sigma}{\partial x_3}]$$

$$\frac{k_3.k_0}{2}\frac{\partial\Sigma}{\partial x_3} \to \frac{\lambda_2 k_1.k_0}{2}\frac{\partial\Sigma}{\partial x_3}$$

They add up to:

$$\lambda_2 k_2.k_0\frac{\partial^2\Sigma}{\partial x_2\partial x_1} = \lambda_2\frac{\partial}{\partial x_2}[\frac{k_1.k_0}{2}\frac{\partial\Sigma}{\partial x_1}]$$

For the above argument to go through in the interacting case we need the following property for the Taylor expansion coefficients:

$$\frac{\partial^2}{\partial x_n\partial x_m}[D_k(x,y)\bar{\Sigma}] = \frac{\partial}{\partial x_{n+m}}[D_k(x,y)\bar{\Sigma}] \tag{4.0.20}$$

(and the same obviously for Y_n). It is demonstrated in the Appendix that this is in fact true.

Thus in general consider:

$$k_n(\sigma_1).k_m(\sigma_2)[\frac{\partial^2 D_k(x,y)\bar{\Sigma}}{\partial x_n\partial y_m}]\mid_{x=y} + k_{n+m}(\sigma_2).k_0(\sigma_1)[\frac{\partial D_k(x,y)\bar{\Sigma}}{\partial y_{n+m}}]\mid_{x=y} \tag{4.0.21}$$

$$\to \int d\sigma\lambda_n(\sigma)k_0(\sigma_1).k_m(\sigma_2)[\frac{\partial^2 D_k(x,y)\bar{\Sigma}}{\partial x_n\partial y_m}]\mid_{x=y} + \int d\sigma\lambda_n(\sigma)k_m(\sigma_2).k_0(\sigma_1)[\frac{\partial D_k(x,y)\bar{\Sigma}}{\partial y_{n+m}}]\mid_{x=y} \tag{4.0.22}$$

$$= \int d\sigma \lambda_n(\sigma) k_0(\sigma_1).k_m(\sigma_2)[(\frac{\partial}{\partial x_n} + \frac{\partial}{\partial y_n})\frac{\partial D_k(x,y)\bar{\Sigma}}{\partial y_m}] \mid_{x=y} \tag{4.0.23}$$

$$= \int d(\sigma)\lambda_n(\sigma)\frac{\partial}{\partial x_n}[k_0(\sigma_1).k_m(\sigma_2)\frac{\partial D_k(x,y)\bar{\Sigma}}{\partial y_m} \mid_{x=y}] \tag{4.0.24}$$

Similarly,

$$k_n(\sigma_2).k_m(\sigma_1)[\frac{\partial^2 D_k(x,y)\bar{\Sigma}}{\partial x_m \partial y_n}] \mid_{x=y} + k_{n+m}(\sigma_1).k_0(\sigma_2)[\frac{\partial D_k(x,y)\bar{\Sigma}}{\partial x_{n+m}}] \mid_{x=y} \tag{4.0.25}$$

$$\to \int d\sigma \lambda_n(\sigma) k_0(\sigma_2).k_m(\sigma_1)[\frac{\partial^2 D_k(x,y)\bar{\Sigma}}{\partial x_m \partial y_n}] \mid_{x=y} + \int d\sigma \lambda_n(\sigma) k_m(\sigma_1).k_0(\sigma_2)[\frac{\partial D_k(x,y)\bar{\Sigma}}{\partial x_{n+m}}] \mid_{x=y} \tag{4.0.26}$$

$$= \int d\sigma \lambda_n(\sigma) k_0(\sigma_2).k_m(\sigma_1)[(\frac{\partial}{\partial x_n} + \frac{\partial}{\partial y_n})\frac{\partial D_k(x,y)\bar{\Sigma}}{\partial x_m}] \mid_{x=y} \tag{4.0.27}$$

$$= \int d(\sigma)\lambda_n(\sigma)\frac{\partial}{\partial x_n}[k_0(\sigma_2).k_m(\sigma_1)\frac{\partial D_k(x,y)\bar{\Sigma}}{\partial x_m} \mid_{x=y}] \tag{4.0.28}$$

Adding the two we find that the λ_n variation is a total derivative in x_n of A even after Taylor expanding.

Similarly the tracelessness constraint of the free theory generalizes to

$$< \int d(\sigma)\lambda_p(\sigma)[k_n(\sigma_1).k_m(\sigma_2)].... >= 0 \tag{4.0.29}$$

in the equation of motion. All the above guarantees that the variation of e^A is a total derivative and therefore the equations of motion obtained by varying w.r.t Σ are invariant.

5 Examples

5.1 Vector k_1 Contribution to Y_1^μ

Our starting point is e^A given by

$$e^{i\{\int d\sigma k_0(\sigma)Y(\sigma) + i\sum_{n>0} k_n(\sigma)\frac{\partial Y(\sigma)}{x_n(\sigma)}\}}$$

$$e^{\int\int d\sigma_1 d\sigma_2\{k_0(\sigma_1)k_0(\sigma_2)[\tilde{\Sigma}+\tilde{G}](\sigma_1,\sigma_2) + (\sum_{n>0} k_n(\sigma_1).k_0(\sigma_2)\frac{\partial[\tilde{\Sigma}+\tilde{G}](\sigma_1,\sigma_2)}{\partial x_n(\sigma_1)} + \sigma_1 \leftrightarrow \sigma_2)\}}$$

$$e^{\int\int d\sigma_1 d\sigma_2\{\sum_{n,m>0} k_n(\sigma_1).k_m(\sigma_2)\frac{\partial^2[\tilde{\Sigma}+\tilde{G}](\sigma_1,\sigma_2)}{\partial x_1(\sigma_1)\partial x_1(\sigma_2)}\}} \tag{5.1.1}$$

We keep terms with one k_1 only. There are three terms that contribute.

$$e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{\Sigma}+\tilde{G}]} k_1(\sigma_1).k_0(\sigma_2)\frac{\partial}{\partial x_1(\sigma_1)}[\tilde{\Sigma}+\tilde{G}]e^{i\int k_0 Y}+e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{\Sigma}+\tilde{G}]}i\int k_1 Y_1 e^{i\int k_0 Y} \tag{5.1.2}$$

In leading order we have:

$$\frac{\partial}{\partial x_1(\sigma_1)}\tilde{\Sigma}(\sigma_1,\sigma_2)=\frac{\partial\bar{\Sigma}}{\partial x_1}=\frac{1}{2}\frac{\partial\Sigma}{\partial x_1}$$

$$\tilde{\Sigma}=\bar{\Sigma}=\Sigma \tag{5.1.3}$$

We get

$$k_1(\sigma_1).k_0(\sigma_2)\frac{\partial\bar{\Sigma}}{\partial x_1}e^{i\int k_0 Y}+k_1(\sigma_2).k_0(\sigma_1)\frac{\partial\bar{\Sigma}}{\partial y_1}e^{i\int k_0 Y}+k_0(\sigma_1).k_0(\sigma_2)\bar{\Sigma}i\int k_1 Y_1 e^{i\int k_0 Y} \tag{5.1.4}$$

which on setting $x_n = y_n$ becomes

$$=\frac{k_1(\sigma_1).k_0(\sigma_2)+k_1(\sigma_2).k_0(\sigma_1)]}{2}\frac{\partial\Sigma}{\partial x_1}e^{i\int k_0 Y}+k_0(\sigma_1).k_0(\sigma_2)\Sigma i\int k_1 Y_1 \tag{5.1.5}$$

Setting $\frac{\delta}{\delta\Sigma}$ of this expression to zero we get the equation (we can set all the σ 's to be equal)

$$-k_1(\sigma_1)k_0(\sigma_1)ik_0^\mu Y_1^\mu + k_0(\sigma_1).k_0(\sigma_1)ik_1^\mu Y_1^\mu = 0 \tag{5.1.6}$$

Converting to space-time fields the coefficient of Y_1^μ is:

$$= -\partial_\mu\partial^\nu A^\mu + \partial_\mu\partial^\mu A^\nu = \partial_\mu F^{\mu\nu} = 0 \tag{5.1.7}$$

which is Maxwell's equation. (5.1.6) is clearly invariant under $k_1(\sigma_1)\rightarrow k_1(\sigma_1)+\int d\sigma\lambda_1(\sigma)k_0(\sigma_1)$, which in terms of space-time fields is $A_\mu \rightarrow A_\mu + \partial_\mu\Lambda$.

5.2 k_1k_1 and k_2 Contribution to Y_1^μ

(i)

$$\frac{1}{2!}\{k_1(\sigma_1).k_0(\sigma_2)\frac{\partial}{\partial x_1(\sigma_1)}[\tilde{\Sigma}+\tilde{G}]+\sigma_1\leftrightarrow\sigma_2\}$$

$$\{k_1(\sigma_3).k_0(\sigma_4)\frac{\partial}{\partial x_1(\sigma_3)}[\tilde{\Sigma}+\tilde{G}]+\sigma_3\leftrightarrow\sigma_4\}e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{\Sigma}+\tilde{G}]}e^{i\int k_0 Y} \tag{5.2.8}$$

(ii)

$$k_1(\sigma_1).k_1(\sigma_2)\frac{\partial^2[\tilde{\Sigma}+\tilde{G}](\sigma_1,\sigma_2)}{\partial x_1(\sigma_1)\partial x_1(\sigma_2)}e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{\Sigma}+\tilde{G}]}e^{i\int k_0 Y} \tag{5.2.9}$$

(iii)

$$e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{\Sigma}+\tilde{G}]}\{k_1(\sigma_1).k_0(\sigma_2)\frac{\partial}{\partial x_1(\sigma_1)}[\tilde{\Sigma}+\tilde{G}]+\sigma_1\leftrightarrow\sigma_2\}i\int k_1 Y_1 e^{i\int k_0 Y} \tag{5.2.10}$$

Let us consider each in turn:

(i)

Using the leading order expressions given in (5.1.3) we get

$$2\times\frac{1}{2!}e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{G}]}k_1(\sigma_1).k_0(\sigma_2)\frac{\partial\Sigma}{\partial x_1}[2k_1(\sigma_3).k_0(\sigma_4)\frac{\partial\tilde{G}(\sigma_3,\sigma_4)}{\partial x_1(\sigma_3)}]e^{i\int k_0 Y} \tag{5.2.11}$$

Varying w.r.t Σ gives

$$=-2e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{G}]}k_1(\sigma_1).k_0(\sigma_2)k_1(\sigma_3).k_0(\sigma_4)\frac{\partial\tilde{G}(\sigma_3,\sigma_4)}{\partial x_1(\sigma_3)}i\int k_0\frac{\partial Y}{\partial x_1}e^{i\int k_0 Y} \tag{5.2.12}$$

Now we have to consider all possible contractions of the k_n's. In order to keep track of the possibilities we separate them into two cases: those involving only one point on the world sheet (i.e. only one vertex operator) and those involving two distinct points (two vertex operators).

One Vertex Operator

In the first case we have to be careful about regularizing. Let us refer to the point as σ_A with $z(\sigma_A)=z$ as the location of the vertex operator. When there is a need for regularizing we will let σ_B be the second point with $z(\sigma_B)-z(\sigma_A)=\epsilon$. We now let the various σ_i be equal to σ_A or σ_B in all possible combinations, but we divide by 2 since these are actually the same point.

Following this procedure we see that regularization is required for σ_3,σ_4 when they stand for the same point (and also for σ_5,σ_6, which we ignore for the moment). Thus we can let

$$\sigma_3=\sigma_A,\ \sigma_4=\sigma_B$$

and σ_1,σ_2 can be anything. This gives

$$-2k_1(\sigma_A).k_0(\sigma_A)k_1(\sigma_A).k_0(\sigma_B)\frac{1}{z_A-z_B} \tag{5.2.13}$$

Now we let $z_A \to z_B$ and $\sigma_A \to \sigma_B$ and use:

$$< k_1^\mu(\sigma_A) k_1^\nu(\sigma_A) >=< k_1^\mu(\sigma_A) k_1^\nu(\sigma_B) >= S_{1,1}^{\mu\nu}(k_0) \tag{5.2.14}$$

This gives

$$2S_{1,1}^{\mu\nu} k_0^\mu k_0^\nu \frac{1}{\epsilon}. \tag{5.2.15}$$

The other possibility is

$$\sigma_3 = \sigma_B, \ \sigma_4 = \sigma_A$$

and again σ_1, σ_2 can be anything, which gives

$$-2k_1(\sigma_A).k_0(\sigma_A)k_1(\sigma_B).k_0(\sigma_A)\frac{1}{z_B - z_A} \tag{5.2.16}$$

Using (5.2.14) gives

$$2S_{1,1}^{\mu\nu} k_0^\mu k_0^\nu \frac{-1}{\epsilon} \tag{5.2.17}$$

Adding the two ((5.2.15) and (5.2.17)) we get zero.

Two Vertex Operators

Now we go to the second possibility viz. there are two distinct points. Let us call them σ_I and σ_{II} and let $z(\sigma_I) = z$ and $z(\sigma_{II}) = w$. Thus we can have a)$\sigma_1 = \sigma_I$ and $\sigma_3 = \sigma_{II}$ or b) vice versa.

Consider a):

First we consider the case that does not require regularization.

Non-singular Case

$$-2k_1(\sigma_I).k_0(\sigma_2)k_1(\sigma_{II}).k_0(\sigma_4)\frac{1}{w - z(\sigma_4)} \tag{5.2.18}$$

Let $\sigma_4 = \sigma_I$ and $\sigma_2 = \sigma_I \ or \ \sigma_{II}$

We now use the notation $k_0(\sigma_I) = p$ and $k_0(\sigma_{II}) = q$. Thus

$$< k_1^\mu(\sigma_I) >= A^\mu(p)$$
$$< k_1^\mu(\sigma_{II}) >= A^\mu(q) \tag{5.2.19}$$

and we get as contribution to the equation of motion:

$$\int dz \int dw A^\mu(p)(p+q)^\mu A^\nu(q) p^\nu \frac{1}{w - z} \tag{5.2.20}$$

We have explicitly written out the integrals over z and w to emphasize the symmetry. Thus by antisymmetry of the integrand in z, w this is zero.

Singular Case

If we let $\sigma_4 = \sigma_{II}$ (and $\sigma_2 = \sigma_I \; or \; \sigma_{II}$) we have to regularize. So we split σ_{II} into σ_A and σ_B as before. Again either

$$\sigma_3 = \sigma_A \; and \; \sigma_4 = \sigma_B$$

which gives

$$-2k_1(\sigma_I).k_0(\sigma_2)k_1(\sigma_A).k_0(\sigma_B)\frac{1}{w_A - w_B} \tag{5.2.21}$$

and using (5.2.19)

$$= A^\mu(p)(p+q)^\mu A^\nu(q)q^\nu(\frac{-1}{\epsilon}) \tag{5.2.22}$$

or

$$\sigma_3 = \sigma_B \; and \; \sigma_4 = \sigma_A$$

which gives

$$-2k_1(\sigma_I).k_0(\sigma_2)k_1(\sigma_B).k_0(\sigma_A)\frac{1}{w_B - w_A} \tag{5.2.23}$$

and using (5.2.19)

$$= A^\mu(p)(p+q)^\mu A^\nu(q)q^\nu(\frac{+1}{\epsilon}) \tag{5.2.24}$$

Adding the two contributions again gives zero.

We have also to look at possibility b) which was $\sigma_1 = \sigma_{II}$ and $\sigma_3 = \sigma_{II}$ Analogous to (5.2.20) one gets

$$\int dz \int dw A^\mu(q)(p+q)^\mu A^\nu(p)p^\nu \frac{1}{z-w} \tag{5.2.25}$$

Note that this is (upto a sign) (5.2.20) with the labels p, q interchanged. But p, q being integration variables we get back (5.2.20) but the overall sign being opposite, they cancel. The integration $\int\int dzdw$ also ensures the vanishing of each term , viz (5.2.20) and (5.2.25), individually. This is also as it should be since interchanging z with w is equivalent to interchanging momenta.

(ii)

$$k_1(\sigma_1).k_1(\sigma_2)\frac{\partial^2\tilde{\Sigma}(\sigma_1,\sigma_2)}{\partial x_1(\sigma_1)\partial x_1(\sigma_2)} e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{G}]} e^{i\int k_0 Y} \tag{5.2.26}$$

Use

$$\frac{\partial^2\tilde{\Sigma}(\sigma_1,\sigma_2)}{\partial x_1(\sigma_1)\partial x_1(\sigma_2)} = \frac{\partial^2\bar{\Sigma}}{\partial x_1\partial y_1} + ...$$

On setting $x_n = y_n$,

$$= \frac{1}{2}(\frac{\partial^2\Sigma}{\partial x_1^2} - \frac{\partial\Sigma}{\partial x_2}) + ... \tag{5.2.27}$$

Only the first term can give, on integration by parts, a contribution to Y_1:

$$e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{G}]}\int k_0(\sigma_7).k_0(\sigma_8)\frac{\partial\tilde{G}(\sigma_7,\sigma_8)}{\partial x_1}k_1(\sigma_1).k_1(\sigma_2)ik_0Y_1e^{i\int k_0Y} \tag{5.2.28}$$

$$\tilde{G}(\sigma_7,\sigma_8) = ln|z(\sigma_7)-z(\sigma_8)| + \frac{\alpha_1}{z(\sigma_7)-z(\sigma_8)} - \frac{\beta_1}{z(\sigma_7)-z(\sigma_8)} + ... \tag{5.2.29}$$

So

$$\frac{\partial\tilde{G}}{\partial x_1} = 0 + \ higher\ order\ in\ x_n$$

We do not get any contribution.

(iii)

$$e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{G}]}k_1(\sigma_1).k_0(\sigma_2)\frac{\partial\tilde{\Sigma}}{\partial x_1(\sigma_1)} + \sigma_1 \leftrightarrow \sigma_2 + \tilde{\Sigma} \leftrightarrow \tilde{G} \tag{5.2.30}$$

Using $\frac{\partial\tilde{G}}{\partial x_1} = 0$ the leading order contribution is zero.

Thus combining (i),(ii) and (iii) we conclude that there are no corrections to $\partial_\mu F^{\mu\nu} = 0$ to this order.

5.3 k_1k_1, k_2 Contributions to Y_2^μ

We start with, as usual, e^A given by

$$e^{i\{\int d\sigma k_0(\sigma)Y(\sigma)+i\sum_{n>0}k_n(\sigma)\frac{\partial Y(\sigma)}{x_n(\sigma)}\}}$$

$$e^{\int\int d\sigma_1 d\sigma_2\{k_0(\sigma_1)k_0(\sigma_2)[\tilde{\Sigma}+\tilde{G}](\sigma_1,\sigma_2)+(\sum_{n>0}k_n(\sigma_1).k_0(\sigma_2)\frac{\partial[\tilde{\Sigma}+\tilde{G}](\sigma_1,\sigma_2)}{\partial x_n(\sigma_1)}+\sigma_1\leftrightarrow\sigma_2)\}}$$

$$e^{\int\int d\sigma_1 d\sigma_2\{\sum_{n,m>0}k_n(\sigma_1).k_m(\sigma_2)\frac{\partial^2[\tilde{\Sigma}+\tilde{G}](\sigma_1,\sigma_2)}{\partial x_1(\sigma_1)\partial x_1(\sigma_2)}\}} \tag{5.3.31}$$

Pick out the terms that contribute to Y_2^μ involving k_1k_1 and k_2.

There are four terms:

(i)

$$e^{\int k_0(\sigma_3).k_0(\sigma_4)[\tilde{G}]}\{k_1(\sigma_1).k_0(\sigma_2)\frac{\partial\tilde{\Sigma}}{\partial x_1(\sigma_1)} + \sigma_1 \leftrightarrow \sigma_2\}i\int k_1 Y_1 e^{i\int k_0 Y} \quad (5.3.32)$$

(ii)

$$e^{\int k_0(\sigma_3).k_0(\sigma_4)[\tilde{G}]}\{k_2(\sigma_1).k_0(\sigma_2)\frac{\partial\tilde{\Sigma}}{\partial x_2(\sigma_1)} + \sigma_1 \leftrightarrow \sigma_2\}e^{i\int k_0 Y} \quad (5.3.33)$$

(iii)

$$e^{\int k_0(\sigma_3).k_0(\sigma_4)[\tilde{G}]}\{k_1(\sigma_1).k_1(\sigma_2)\frac{\partial^2[\tilde{\Sigma}+\tilde{G}](\sigma_1,\sigma_2)}{\partial x_1(\sigma_1)\partial x_1(\sigma_2)}\}e^{i\int k_0 Y} \quad (5.3.34)$$

(iv)

$$e^{\int k_0(\sigma_3).k_0(\sigma_4)[\tilde{G}+\tilde{\Sigma}]}i\int k_2 Y_2 e^{i\int k_0 Y} \quad (5.3.35)$$

We expand $\tilde{\Sigma}$ using (5.1.3) and vary w.r.t. Σ:

(i)

$$e^{\int k_0(\sigma_3).k_0(\sigma_4)[\tilde{G}]}\{k_1(\sigma_1).k_0(\sigma_2)\frac{\partial\bar{\Sigma}}{\partial x_1} + \sigma_1 \leftrightarrow \sigma_2\}i\int k_1 Y_1 e^{i\int k_0 Y}$$

$$= e^{\int k_0(\sigma_3).k_0(\sigma_4)[\tilde{G}]}k_1(\sigma_1).k_0(\sigma_2)\frac{\partial\Sigma}{\partial x_1}i\int k_1(\sigma)Y_1 e^{i\int k_0 Y}$$

$\frac{\delta}{\delta\Sigma}$ gives:

$$-e^{\int k_0(\sigma_3).k_0(\sigma_4)[\tilde{G}]}k_1(\sigma_1).k_0(\sigma_2)i\int k_1(\sigma)Y_2 e^{i\int k_0 Y} \quad (5.3.36)$$

We have to make contractions of the k_n's. Let $z(\sigma_1) = z$ and assign the momentum p to this point. Let $z(\sigma) = w$ and assign momentum q. Now $\sigma_2 = \sigma_1$ or $\sigma_2 = \sigma$ are two possibilities and for each of these we can have $\sigma_3 = \sigma_1$, $\sigma_4 = \sigma$ or $\sigma_3 = \sigma$, $\sigma_4 = \sigma_1$. None of the above need regularization. We can also include the following two possibilities that need a regulator:$\sigma_3 = \sigma_4 = \sigma$ or $\sigma_3 = \sigma_4 = \sigma_1$. For these cases we will let σ_A and σ_B be the "point splitting" of σ . Thus ($\sigma_3 = \sigma_A$ and $\sigma_4 = \sigma_B$) or ($\sigma_3 = \sigma_B$ and $\sigma_4 = \sigma_A$). We will weight these with a factor of 1/2. This gives $q^2\ ln\ \epsilon$. Similarly point splitting σ_1 gives $p^2\ ln\ \epsilon$. Putting all the above together and using (5.2.19)we get:

$$-A(p).(p+q)iA^\mu(q)|z-w|^{2p.q}(\epsilon)^{p^2+q^2}Y_2^\mu e^{i(p+q)Y} \quad (5.3.37)$$

If we multiply and divide by $\epsilon^{2p.q}$ this becomes

$$-A(p).(p+q)iA^{\mu}(q)|\frac{z-w}{\epsilon}|^{2p.q}(\epsilon)^{(p+q)^2}Y_2^{\mu}e^{i(p+q)Y} \tag{5.3.38}$$

Interchanging the role of σ and σ_1 i.e. setting $z(\sigma_1)=w$ and $z(\sigma)=z$, we get the same expression with p,q interchanged. Since these are just dummy variables we can combine the two if we allow both p and q to vary over the full range of values.

$$-A(p).(p+q)iA^{\mu}(q)|\frac{z-w}{\epsilon}|^{2p.q}(\epsilon)^{(p+q)^2}Y_2^{\mu}e^{i(p+q)Y} \tag{5.3.39}$$

However

There is also the possibility that $\sigma=\sigma_1$. In this case we point split and let $\sigma=\sigma_A$,$\sigma_1=\sigma_B$ or vice versa (with weight 1/2 to each). Then using (5.2.14) we get

$$-S_{1,1}^{\mu\nu}k_0^{\nu}iY_2^{\mu}e^{i\int k_0Y}(\epsilon)^{k_0^2} \tag{5.3.40}$$

(ii)

$$e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{G}]}\{k_2(\sigma_1).k_0(\sigma_2)\frac{\partial\tilde{\Sigma}}{\partial x_2(\sigma_1)}+\sigma_1\leftrightarrow\sigma_2\}e^{i\int k_0Y} \tag{5.3.41}$$

Using the approximations:

$$\frac{\partial}{\partial x_2(\sigma_1)}\tilde{\Sigma}(\sigma_1,\sigma_2)\approx\frac{\partial\bar{\Sigma}}{\partial x_2}=\frac{1}{2}\frac{\partial\Sigma}{\partial x_2}$$

$$\tilde{\Sigma}\approx\bar{\Sigma}=\Sigma \tag{5.3.42}$$

$$e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{G}]}k_2(\sigma_1).k_0(\sigma_2)\frac{\partial\Sigma}{\partial x_2}e^{i\int k_0Y} \tag{5.3.43}$$

$\frac{\delta}{\delta\Sigma}$ gives

$$-e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{G}]}k_2(\sigma_1).k_0(\sigma_2)ik_0^{\mu}Y_2^{\mu}e^{i\int k_0Y} \tag{5.3.44}$$

$$=-(\epsilon)^{k_0^2}S_2(k_0).k_0ik_0^{\mu}Y_2^{\mu} \tag{5.3.45}$$

(iii)

$$e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{G}]}k_1(\sigma_1).k_1(\sigma_2)\frac{\partial^2\tilde{\Sigma}}{\partial x_1(\sigma_1)\partial x_1(\sigma_2)} \tag{5.3.46}$$

$$\frac{\partial^2 \tilde{\Sigma}}{\partial x_1(\sigma_1)\partial x_1(\sigma_2)} \approx \frac{1}{2}(\frac{\partial^2\Sigma}{\partial x_1^2} - \frac{\partial \Sigma}{\partial x_2}) + ... \tag{5.3.47}$$

$$= e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{G}]} k_1(\sigma_1).k_1(\sigma_2)\frac{1}{2}(\frac{\partial^2\Sigma}{\partial x_1^2} - \frac{\partial \Sigma}{\partial x_2})e^{i\int k_0 Y} \tag{5.3.48}$$

$\frac{\delta}{\delta\Sigma}$

$$= e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{G}]} k_1(\sigma_1).k_1(\sigma_2) i k_0 Y_2 e^{i\int k_0 Y} \tag{5.3.49}$$

Following the procedures described earlier this gives (when $\sigma_1 \neq \sigma_2$):

$$(\epsilon)^{(p+q)^2}|\frac{z-w}{\epsilon}|^{2p.q} A(p).A(q) i(p+q)^\mu Y_2^\mu e^{i\int k_0 Y} \tag{5.3.50}$$

Both p and q are integrated over the entire range.

When $\sigma_1 = \sigma_2$ one has to point split and this gives:

$$-(\epsilon)^{k_0^2} i S_2^\mu(k_0) Y_2^\mu e^{i\int k_0 Y} \tag{5.3.51}$$

(iv)

$$= e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{\Sigma}+\tilde{G}]} i k_2^\nu Y_2^\mu \tag{5.3.52}$$

Using $\tilde{\Sigma} \approx \Sigma$ and varying w.r.t.Σ gives

$$= e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{G}]} k_0(\sigma_1).k_0(\sigma_2) i k_2^\nu Y_2^\mu e^{i\int k_0 Y} \tag{5.3.53}$$

$$= -(\epsilon)^{k_0^2} k_0^2 i S_2^\mu(k_0) Y_2^\mu e^{i\int k_0 Y} \tag{5.3.54}$$

We can also check that the equations are invariant at the loop variable level:

$$\begin{aligned}
\delta(i) &= -\lambda_1(\sigma)k_0(\sigma_1).k_0(\sigma_2) i k_1 Y_2 e^{i\int k_0 Y} e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{G}]} && (5.3.55)\\
&\quad -\lambda_1(\sigma)k_1(\sigma_1).k_0(\sigma_2) i k_0 Y_2 e^{i\int k_0 Y} e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{G}]} \\
\delta(ii) &= -\lambda_1(\sigma)k_1(\sigma_1).k_0(\sigma_2) i k_0 Y_2 e^{i\int k_0 Y} e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{G}]} && (5.3.56)\\
&\quad -\lambda_2(\sigma)k_0(\sigma_1).k_0(\sigma_2) i k_0 Y_2 e^{i\int k_0 Y} e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{G}]} \\
\delta(iii) &= 2\lambda_1(\sigma)k_1(\sigma_1).k_0(\sigma_2) i k_0 Y_2 e^{i\int k_0 Y} e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{G}]} && (5.3.57)\\
\delta(iv) &= \lambda_1(\sigma)k_0(\sigma_1).k_0(\sigma_2) i k_1 Y_2 e^{i\int k_0 Y} e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{G}]} && (5.3.58)\\
&\quad +\lambda_2(\sigma)k_0(\sigma_1).k_0(\sigma_2) i k_0 Y_2 e^{i\int k_0 Y} e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{G}]}
\end{aligned}$$

Clearly the variations add up to zero. In Section 6 we will discuss the gauge transformation law for space-time fields.

5.4 $k_1k_1k_1, k_2k_1, k_3$ Contribution to Y_1^μ

There are many terms that contribute. We will consider only the following term to illustrate the technique being used.

(i)

$$\frac{1}{2!}e^{k_0(\sigma_7).k_0(\sigma_8)\tilde{G}}[k_1(\sigma_1).k_0(\sigma_2)\frac{\partial\tilde{\Sigma}}{\partial x_1(\sigma_1)}+\sigma_1\leftrightarrow\sigma_2]$$

$$[k_1(\sigma_3).k_0(\sigma_4)\frac{\partial\tilde{G}}{\partial x_1(\sigma_3)}+\sigma_3\leftrightarrow\sigma_4][k_1(\sigma_5).k_0(\sigma_6)\frac{\partial\tilde{G}}{\partial x_1(\sigma_5)}+\sigma_5\leftrightarrow\sigma_6]e^{i\int k_0Y}=0. \tag{5.4.59}$$

We start with contractions that do not involve any regularization ("non singular case"). We will treat the cases that require regularization ("singular case") separately. In each case depending on the number of distinct points we have different contributions.

Non Singular Case:

Three Vertex Operators:

We first consider contractions involving three distinct verex operators. Let us designate σ_I, σ_{II} and σ_{III} as the labels of the vertex operators with locations $z(\sigma_I)=u, z(\sigma_{II})=w, z(\sigma_{III})=z$ and momenta p,q and k respectively being associated with these vertex operators.

We use the approximation that $\tilde{\Sigma}\approx\Sigma$ as before and integrate by parts on x_1 to get:

$$\frac{1}{2!}e^{k_0(\sigma_7).k_0(\sigma_8)\tilde{G}}[k_1(\sigma_1).k_0(\sigma_2)\frac{\partial\Sigma}{\partial x_1(\sigma_1)}+\sigma_1\leftrightarrow\sigma_2]$$

$$[k_1(\sigma_3).k_0(\sigma_4)\frac{\partial\tilde{G}}{\partial x_1(\sigma_3)}+\sigma_3\leftrightarrow\sigma_4][k_1(\sigma_5).k_0(\sigma_6)\frac{\partial\tilde{G}}{\partial x_1(\sigma_5)}+\sigma_5\leftrightarrow\sigma 6]e^{i\int k_0Y} \tag{5.4.60}$$

$$=-\frac{1}{2!}e^{k_0(\sigma_7).k_0(\sigma_8)\tilde{G}}k_1(\sigma_1).k_0(\sigma_2)[k_1(\sigma_3).k_0(\sigma_4)\frac{\partial\tilde{G}}{\partial x_1(\sigma_3)}+\sigma_3\leftrightarrow\sigma_4]$$

$$[k_1(\sigma_5).k_0(\sigma_6)\frac{\partial\tilde{G}}{\partial x_1(\sigma_5)}+\sigma_5\leftrightarrow\sigma_6]ik_0Y_1e^{i\int k_0Y} \tag{5.4.61}$$

Let us first consider contractions that do not involve regularization. Thus consider the assignments

$$\begin{array}{lll} p\leftrightarrow z(\sigma_I)=u & \sigma_1=\sigma_I & \sigma_2=\sigma_I,\sigma_{II},\sigma_{III} \\ q\leftrightarrow z(\sigma_{II})=w & \sigma_3=\sigma_{II} & \sigma_4=\sigma_I,\sigma_{III} \\ k\leftrightarrow z(\sigma_{III})=z & \sigma_5=\sigma_{III} & \sigma_6=\sigma_I,\sigma_{II} \end{array} \tag{5.4.62}$$

This gives (using (5.2.19)

$$|\frac{z-w}{\epsilon}|^{2k.q}|\frac{w-u}{\epsilon}|^{2p.q}|\frac{z-w}{\epsilon}|^{2p.k}(\epsilon)^{(p+q+k)^2}$$

$$A(p).(p+q+k)[\frac{A(q).p}{w-u}+\frac{A(q).k}{w-z}][\frac{A(k).p}{z-u}+\frac{A(k).q}{z-w}]i(p+q+k)^{\mu}Y_1^{\mu}e^{i(p+q+k)} \tag{5.4.63}$$

Two Vertex Operators

Now we come to the case where there are two vertex operators. The assignment that does not need regularization is:

(I)

$$\begin{aligned} p \leftrightarrow z(\sigma_I) = w \quad & \sigma_1 = \sigma_I \quad \sigma_2 = \sigma_I, \sigma_{II} \\ & \sigma_3 = \sigma_I \quad \sigma_4 = \sigma_{II} \\ q \leftrightarrow z(\sigma_{II}) = z \quad & \sigma_5 = \sigma_{II} \quad \sigma_6 = \sigma_I \end{aligned} \tag{5.4.64}$$

This gives:

$$|\frac{z-w}{\epsilon}|^{2p.q}(\epsilon)^{(p+q)^2}S_{1,1}^{\mu\nu}(p)(p+q)^{\mu}q^{\nu}\frac{1}{w-z}A^{\rho}(q)p^{\rho}\frac{1}{z-w}i(p+q)^{\sigma}Y_1^{\sigma}e^{i(p+q)Y} \tag{5.4.65}$$

The other possible assignment is:

(II)

$$\begin{aligned} p \leftrightarrow z(\sigma_I) = w \quad & \sigma_1 = \sigma_I \quad \sigma_2 = \sigma_I, \sigma_{II} \\ q \leftrightarrow z(\sigma_{II}) = z \quad & \sigma_3 = \sigma_{II} \quad \sigma_4 = \sigma_I \\ & \sigma_5 = \sigma_{II} \quad \sigma_6 = \sigma_I \end{aligned} \tag{5.4.66}$$

which gives

$$|\frac{z-w}{\epsilon}|^{2p.q}(\epsilon)^{(p+q)^2}A^{\mu}(p)(p+q)^{\mu}S_{1,1}^{\nu\rho}(q)\frac{p^{\nu}p^{\rho}}{(w-z)^2}i(p+q)^{\sigma}Y_1^{\sigma}e^{i(p+q)Y} \tag{5.4.67}$$

Singular Cases:

Three Vertex Operators:

We now consider assignments that require regularization. For the three vertex operator case we have:

$$\begin{aligned} p \leftrightarrow z(\sigma_I) = u \quad & \sigma_1 = \sigma_I \quad \sigma_2 = \sigma_I, \sigma_{II}, \sigma_{III} \\ q \leftrightarrow z(\sigma_{II}) = w \quad & \sigma_3 = \sigma_{II_A} \quad \sigma_4 = \sigma_{II_B}, \\ k \leftrightarrow z(\sigma_{III}) = z \quad & \sigma_5 = \sigma_{III} \quad \sigma_6 = \sigma_I, \sigma_{II} \end{aligned} \tag{5.4.68}$$

$$|\frac{z-w}{\epsilon}|^{2k.q}|\frac{w-u}{\epsilon}|^{2p.q}|\frac{z-w}{\epsilon}|^{2p.k}(\epsilon)^{(p+q+k)^2}k_1(\sigma_A).[k_0(\sigma_I)+k_0(\sigma_{II})+k_0(\sigma_{III})]$$
$$\underbrace{[\frac{k_1(\sigma_A).k_0(\sigma_B)}{w_A-w_B}+\frac{k_1(\sigma_B).k_0(\sigma_A)}{w_B-w_A}]}_{=0}([\frac{k_1(\sigma_{III}.k_0(\sigma_I)}{z-u}+\frac{k_1(\sigma_{III}.k_0(\sigma_I)}{z-w}] \tag{5.4.69}$$

We have used $\sigma_A=\sigma_{II_A}$ and $\sigma_B=\sigma_{II_B}$. In the limit $\sigma_A\to\sigma_B$, $<k_1(\sigma_A)>=<k_1(\sigma_B)>=A(q)$ and $<k_0(\sigma_A)>=<k_0(\sigma_B)>=q$ which is why the expression in square brackets vanishes.

Two Vertex Operators:

We turn to the case with two vertex operators.

(I)

$$\begin{array}{lll} p\leftrightarrow z(\sigma_I)=w & \sigma_1=\sigma_{I_A} & \sigma_2=\sigma_I,\sigma_{II} \\ & \sigma_3=\sigma_{I_B} & \sigma_4=\sigma_{I_A} \\ q\leftrightarrow z(\sigma_{II})=z & \sigma_5=\sigma_{II} & \sigma_6=\sigma_I \end{array} \tag{5.4.70}$$

(We also have to consider the assignment with σ_A and σ_B interchanged.)

This gives:

$$k_1(\sigma_A).[k_0(\sigma_A)+k_0(\sigma_{II})][\frac{k_1(\sigma_B).k_0(\sigma_A)}{w_B-w_A}(+\sigma_3\leftrightarrow\sigma_4\ \textit{is not allowed})][\frac{k_1(\sigma_{II}).k_0(\sigma_I)}{z-w}] \tag{5.4.71}$$

(As before we are using $\sigma_{A,B}$ to denote $\sigma_{I_{A,B}}$.).

σ_3 cannot be set to σ_A because $\sigma_1=\sigma_A$. This is why the exchange term involving σ_3 and σ_4 is not allowed.

If we interchange $A\leftrightarrow B$ in (5.4.71) we find

$$k_1(\sigma_B).[k_0(\sigma_B)+k_0(\sigma_{II})][\frac{k_1(\sigma_A).k_0(\sigma_B)}{w_A-w_B}(+\sigma_3\leftrightarrow\sigma_4\ \textit{is not allowed})][\frac{k_1(\sigma_{II}).k_0(\sigma_I)}{z-w}] \tag{5.4.72}$$

In the limit $\sigma_A\to\sigma_B$,

$<k_1^\mu(\sigma_A)k_1^\nu(\sigma_B)>=S_{1,1}^{\mu\nu}$ and $k_0(\sigma_A)=k_0(\sigma_B)=k_0(\sigma_I)=p$ and thus (5.4.71) and (5.4.72) add up to zero.

II

$$\begin{array}{lll} p\leftrightarrow z(\sigma_I)=w & \sigma_1=\sigma_I & \sigma_2=\sigma_I,\sigma_{II} \\ q\leftrightarrow z(\sigma_{II})=z & \sigma_3=\sigma_{II_A} & \sigma_4=\sigma_{II_B} \\ & \sigma_5=\sigma_{II_B} & \sigma_6=\sigma_{II_A} \end{array} \tag{5.4.73}$$

This gives:

$$k_1(\sigma_I).[k_0(\sigma_I)+k_0(\sigma_{II})][\frac{k_1(\sigma_A).k_0(\sigma_B)}{z_A-z_B}\frac{k_1(\sigma_B).k_0(\sigma_A)}{z_B-z_A}]i(p+q)^\mu Y_1^\mu e^{i(p+q)Y}$$

$$+k_1(\sigma_I).[k_0(\sigma_I)+k_0(\sigma_{II})][\frac{k_1(\sigma_B).k_0(\sigma_A)}{z_B-z_A}\frac{k_1(\sigma_A).k_0(\sigma_B)}{z_A-z_B}]i(p+q)^\mu Y_1^\mu e^{i(p+q)Y} \tag{5.4.74}$$

We have used the same shorthand notation as in previous examples. Note that just as in previous cases interchanging σ_3 and σ_4 is not allowed. Note also that the two terms do not cancel. They add to give:

$$-A(p).(p+q)\frac{S_{1,1}^{\mu\nu}(q)q^\mu q^\nu}{\epsilon^2} \tag{5.4.75}$$

We have weighted it by a factor $\frac{1}{2}$ as in (**??**).

The final result is

$$-A(p).(p+q)\frac{S_{1,1}^{\mu\nu}(q)q^\mu q^\nu}{\epsilon^2}|\frac{z-w}{\epsilon}|^{2p.q}(\epsilon)^{(p+q)^2} \tag{5.4.76}$$

Integrals over w and p,q are implicit.

One Vertex Operator

Finally we have to consider the assignment where there is only one vertex operator and this clearly is singular and needs regularization. We will also observe a serious dependence on the prescription. This is not necessarily unacceptable. Presumably different prescriptions involve field redefinitions. If we impose physical state conditions on the fields these dependences should disappear.

$$\begin{array}{lll} z(\sigma_A)=z_A & \sigma_1=\sigma_A & \sigma_2=\sigma \\ z(\sigma_B)=z_B & \sigma_3=\sigma_B & \sigma_4=\sigma_A,\sigma_C \\ z(\sigma_C)=z_C & \sigma_5=\sigma_C & \sigma_6=\sigma_A,\sigma_B \end{array} \tag{5.4.77}$$

We assign the momentum k to the vertex operator. There are a total of 3! ways of assigning labels. So we weight each possibility by $\frac{1}{3!}$. What is given above is only one of the possibilities.

It gives

$$k_1(\sigma_A).[k_0(\sigma_A)+k_0(\sigma_B)+k_0(\sigma_C)][\frac{k_1(\sigma_B).k_0(\sigma_A)}{z_B-z_A}+\frac{k_1(\sigma_B).k_0(\sigma_C)}{z_B-z_C}]$$

$$[\frac{k_1(\sigma_C).k_0(\sigma_B)}{z_C - z_B} + \frac{k_1(\sigma_C).k_0(\sigma_A)}{z_C - z_A}] \tag{5.4.78}$$

Now we take the three points to be equidistant (this is a prescription) and this implies that

$$\begin{aligned} z_B - z_A &= \epsilon \\ z_C - z_B &= \epsilon \\ z_C - z_A &= 2\epsilon \end{aligned} \tag{5.4.79}$$

The expression in the second pair of square brackets vanishes and this term is zero. The term obtained by interchanging σ_A and σ_C also vanishes. Similarly the two terms that have $\sigma_5 = \sigma_B$ also vanish. Thus four of the six possibilities give zero. The remaining two are given by the assignment

$$\begin{aligned} &z(\sigma_A) = z_A \quad \sigma_1 = \sigma_B \quad \sigma_2 = \sigma \\ &z(\sigma_B) = z_B \quad \sigma_3 = \sigma_A \quad \sigma_4 = \sigma_A, \sigma_C \\ &z(\sigma_C) = z_C \quad \sigma_5 = \sigma_C \quad \sigma_6 = \sigma_A, \sigma_B \end{aligned} \tag{5.4.80}$$

and the one obtained by interchanging σ_A and σ_C in this. This gives:

$$k_1(\sigma_B).[k_0(\sigma_A) + k_0(\sigma_B) + k_0(\sigma_C)][\frac{k_1(\sigma_A).k_0(\sigma_B)}{z_A - z_B} + \frac{k_1(\sigma_A).k_0(\sigma_C)}{z_A - z_C}]$$

$$[\frac{k_1(\sigma_C).k_0(\sigma_A)}{z_C - z_A} + \frac{k_1(\sigma_C).k_0(\sigma_B)}{z_C - z_B}] \tag{5.4.81}$$

Plugging the space time fields and the rest of the factors we get

$$S^{\mu\nu\rho}_{1,1,1}(k_0)k_0^\mu[\frac{k_0^\nu}{(-\epsilon)} + \frac{k_0^\nu}{(-2\epsilon)}][\frac{k_0^\rho}{(2\epsilon)} + \frac{k_0^\rho}{(\epsilon)}](\epsilon)^{k_0^2} i k_0^\sigma Y_1^\sigma \tag{5.4.82}$$

$$= -\frac{3}{4} S^{\mu\nu\rho}_{1,1,1} k_0^\mu k_0^\nu k_0^\rho (\epsilon)^{k_0^2 - 2} i k_0^\sigma Y_1^\sigma \tag{5.4.83}$$

We have multiplied the answer by $\frac{2}{3!}$ as the weight for this term.

We have thus calculated the contribution from the first term to the the equation of motion.

What is to be noted is that the field $S_{1,1,1}$ is present as a result of the fact that we did not throw away the singular (normal ordering) pieces. This term will be indispensable in defining gauge transformations because there will be terms that cannot be assigned to any other field - in fact the presence of this term therefore guarantees that a gauge transformation can always be defined.

6 Space-Time Fields and their Transformations

Now we proceed to define fields. In the first approximation they were defined [2, 4] by the following equations:

$$
\begin{aligned}
< k_1^\mu > &= A_1^\mu \\
< k_1^\mu(\sigma_1) k_1^\nu(\sigma_2) > &= \frac{D(\sigma_1 - \sigma_2)}{a} S_{1,1}^{\mu\nu} + A_1^\mu A_1^\nu \\
< k_2^\nu > &= S_2^\mu
\end{aligned}
\tag{6.0.1}
$$

We will define the gauge transformation laws for the space time fields by comparing the variations of the loop variable expression with the field expression. Thus consider expression (i)in Section 5.4 in both forms:

A(i) given in (5.3.37) and (5.3.40):

$$
- A(p).(p+q) i A^\mu(q) |\frac{z-w}{\epsilon}|^{2p.q} (\epsilon)^{(p+q)^2} Y_2^\mu e^{i(p+q)Y} - S_{1,1}^{\mu\nu} k_0^\nu i Y_2^\mu e^{i\int k_0 Y} (\epsilon)^{k_0^2}
\tag{6.0.2}
$$

B(i) given in (5.3.36):

$$
- e^{\int k_0(\sigma_5).k_0(\sigma_6)[\tilde{G}]} k_1(\sigma_1).k_0(\sigma_2) i k_1(\sigma) Y_2 e^{i\int k_0 Y}
\tag{6.0.3}
$$

Integrals over z, w and p, q, k are implicit in all the above. Thus the integral

$$
\int dz \int dw |\frac{z-w}{\epsilon}|^{2p.q} = \int dz F(p,q)
\tag{6.0.4}
$$

$F(p,q)$ is defined only after suitable regularization. The actual evaluation of this function will be done later.

Now we consider the variation of B(i):

$$
\begin{aligned}
\delta B(i) &= [-i\lambda_1(\sigma) k_0(\sigma_1).k_0(\sigma_2) k_1^\mu(\sigma_3) Y_2^\mu \qquad (6.0.5) \\
&\quad -i\lambda_1(\sigma) k_1(\sigma_1).k_0(\sigma_2) i k_0^\mu(\sigma_3) Y_2^\mu] e^{k_0(\sigma_5).k_0(\sigma_6)\tilde{G}} e^{i\int k_0 Y}
\end{aligned}
$$

We convert this to space -time fields :

$$
\begin{aligned}
\delta B(i) &= [-i\Lambda_{1,1}^\mu(k) k_0^2 (\epsilon)^{k_0^2} e^{ik_0 Y} Y_2^\mu \\
&\quad -i\Lambda_1(p) A_1^\mu(q) (p+q)^2 |\frac{z-w}{\epsilon}|^{2p.q} \epsilon^{(p+q)^2} e^{i(p+q)Y} Y_2^\mu] \\
&\quad +[-i\Lambda_{1,1}^\nu(k) k_0^\nu k_0^\mu (\epsilon)^{k_0^2} e^{ik_0 Y} Y_2^\mu \qquad (6.0.6) \\
&\quad -i\Lambda_1(q) A_1^\nu(p) (p+q)^\nu (p+q)^\mu |\frac{z-w}{\epsilon}|^{2p.q} (\epsilon)^{(p+q)^2} e^{i(p+q)} Y_2^\mu]
\end{aligned}
$$

This is to be compared with $\delta A(i)$: We will write

$$\begin{aligned} \delta S^{\mu\nu}_{1,1}(k)e^{ik_0Y} &= [\Lambda^{\mu}_{1,1}(k)k_0^{\nu} + \Lambda^{\nu}_{1,1}(k)k_0^{\mu} + \delta_{int}S^{\mu\nu}_{1,1}] \\ \delta A^{\mu}_1(p)e^{ip_0Y} &= p^{\mu}\Lambda_1(p)e^{ip_0Y} \end{aligned} \quad (6.0.7)$$

and determine $\delta_{int}S^{\mu\nu}_{1,1}$

We get

$$\delta B(i) = \delta_{free}A(i)+$$

$$+\int dpdq\delta(p+q-k)[-iq.(p+q)\Lambda_1(p)A^{\mu}_1(q) - i\Lambda_1(q)p^{\mu}A(p).(p+q)]F(p,q)(\epsilon)^{(p+q)^2}e^{i(p+q)Y}+$$

$$= \delta_{free}A(i) + \delta_{int}S^{\mu\nu}_{1,1}e^{ik_0Y}(\epsilon)^{k_0^2}k_0^{\nu} \quad (6.0.8)$$

This fixes $\delta_{int}S^{\mu\nu}_{1,1}$ to be

$$\delta_{int}S^{\mu\nu}_{1,1}(k) = \int dpdq\delta(p+q-k)[-i\Lambda_1(p)q^{\nu}A^{\mu}_1(q) - i\Lambda_1(q)p^{\mu}A_1(p)^{\nu}]F(p,q) \quad (6.0.9)$$

B(ii) (5.3.44)

$$e^{k_0(\sigma_5).k_0(\sigma_6)\tilde{G}}k_2(\sigma_1).k_0(\sigma_2)ik_0^{\mu}(\sigma_3)e^{ik_0Y}Y_2^{\mu}$$

$$\delta B(ii) = -e^{k_0(\sigma_5).k_0(\sigma_6)\tilde{G}}[\lambda_1(\sigma)k_1(\sigma_1).k_0(\sigma_2)+\lambda_2(\sigma)k_0(\sigma_1).k_0(\sigma_2)]ik_0^{\mu}(\sigma)e^{ik_0Y}Y_2^{\mu}$$

$$= -\Lambda^{\mu}_{1,1}(k_0)k_0^{\nu}ik_0^{\mu}(\epsilon)^{k_0^2}e^{ik_0Y}$$

$$-\int dpdq\delta(p+q-k)\int dw\Lambda_1(p)A^{\nu}_1(q)(p_0+q_0)^{\nu}i(p_0+q_0)^{\mu}|\frac{z-w}{\epsilon}|^{2p.q}(\epsilon)^{(p+q)^2}e^{i(p+q)Y}Y_2^{\mu}$$

$$-\Lambda_2(k_0)k_0^2ik_0^{\mu}(\epsilon)^{k_0^2}Y_2^{\mu}e^{ik_0Y} \quad (6.0.10)$$

A(ii) (5.3.45)

$$A(ii) = -S_2^{\nu}(k_0^{\nu})ik_0(\epsilon)^{k_0^2}Y_2^{\mu}e^{ik_0Y}$$

$$\delta S_2^{\mu}(k_0) = \Lambda_2(k_0)k_0^{\mu} + \Lambda^{\mu}_{1,1}(k_0) + \delta_{int}S_2^{\mu} \quad (6.0.11)$$

$$\begin{aligned} \delta A(ii) = &-\Lambda_2(k_0)k_0^2ik_0^{\mu}(\epsilon)^{k_0^2}Y_2^{\mu}e^{ik_0Y} \\ &-\Lambda^{\mu}_{1,1}(k_0)k_0^{\nu}ik_0^{\mu}(\epsilon)^{k_0^2}e^{ik_0Y} \\ &-\delta_{int}S_2^{\mu}k_0^{\nu}ik_0^{\mu}(\epsilon)^{k_0^2}Y_2^{\mu}e^{ik_0Y} \end{aligned} \quad (6.0.12)$$

Comparing (6.0.10) with (6.0.12) we find

$$\delta B(ii) = \delta A(ii) + \delta S^{\mu}_{2\,int} k^{\nu}_0 i k^{\mu}_0 (\epsilon)^{k_0^2} Y^{\mu}_2 e^{ik_0 Y} -$$
$$\int dpdq\delta(p+q-k)\Lambda_1(p)A^{\nu}_1(q)(p_0+q_0)^{\nu} i(p_0+q_0)^{\mu} \underbrace{\int dw|\frac{z-w}{\epsilon}|^{2p.q}}_{F(p,q)}(\epsilon)^{(p+q)^2} e^{i(p+q)Y} Y^{\mu}_2 \tag{6.0.13}$$

From this we conclude that

$$\delta_{int} S^{\mu}_2(k_0) = \int dpdq\delta(p+q-k)\Lambda_1(p)A^{\nu}_1(q)F(p,q) \tag{6.0.14}$$

Thus we obtain the transformation rules for $S_{1,1}$ and S_2. Equations (iii) and (iv) are clearly consistent with this since they differ only in index structure.

It is not particularly illuminating to describe in detail the gauge transformation law for $S_{1,1,1}$ that one obtains in this manner since the calculation is very similar to that of $S_{1,1}$.

7 Consistency of Gauge Transformations and x_n-dependence of Fields

We examine, in this section, the question of *consistency* of gauge transformations of space-time fields defined in earlier sections. The question arises because there are different equations that can be used to define the gauge transformation law of $S_{1,1}$. For instance when one integrates by parts on x_1, different vertex operators such as Y_1 or Y_2 are obtained depending on whether one differentiates $e^{ik_0 Y}$ twice or acts once each on $e^{ik_0 Y}$ and $e^{k_0 . k_0 \tilde{G}}$. The dependence on $z-w$ is thus different and one obtains instead of $F(p,q)$ (in (6.0.9)) some other function, and thus a different transformation law.

In fact $F(p,q)$ is a function of x_n because $\tilde{G}(z-w) = ln\,(z-w) + O(x_n)$. Thus in principle one can ask what the result of differentiating (6.0.9) by x_n is. The RHS of (6.0.9) is non-zero on differentiating and one reaches an inconsistency unless one assumes that the LHS also is non-zero - i.e. it must be a function of x_n as well. This leads inexorably to the conclusion that the space-time fields such as $S_{1,1}$ must be functions of x_n, $S_{1,1}(k_0, x_n)$.

In the equation defining S (5.2.14) there is a natural way to introduce this dependence, and this is to make the "string field" Ψ a function of x_n. Thus:

$$\int dk_n d\lambda_n k^{\mu}_1 k^{\nu}_1 \Psi[k_n, k_0, \lambda_n, x_n] = S^{\mu\nu}_{1,1}(k_0, x_n) \tag{7.0.1}$$

Since λ_n is to lowest order the shift in x_n, we can change variables [1] to y_n, defined by

$$\sum_n \lambda_n t^{-n} = e^{\sum_n t^{-n} y_n}$$

and replace (7.0.1) by

$$\int dk_n dy_n k_1^\mu k_1^\nu \Psi[k_n, k_0, y_n, x_n] = S_{1,1}^{\mu\nu}(k_0, x_n) \tag{7.0.2}$$

Both x_n and y_n are gauge coordinates. It is necessary therefore to understand the presence of both of these coordinates in the Ψ. Our starting point is a field $\Psi(X(z), x_n + y_n)$. The breakup of the gauge coordinate into $x_n + y_n$ is similar in spirit to that in the background field method in field theory. We treat x_n as a background or reference point. Now we do a generalized Fourier transform using the loop variable and define the variable Y which has in it only x_n. Thus the relation between the original string field and the one we have been using in this paper can be summarized in the following way: (We use the symbol Ψ for all the fields - the arguments of the fields will make clear which field we are referring to)

$$\Psi(X, x_n + y_n) = \Psi(Y_n, x_n, y_n) = \int [dk_n] \Psi(k_n, x_n, y_n) e^{i \sum_n k_n Y_n}$$

Thus while the original field is only a function of $x_n + y_n$, once we define the variable Y we have specified a reference point. The space-time fields obtained by (5.2.14) thus depend on this reference point. Gauge invariance is the statement that physics is independent of $x_n + y_n$. In terms of the new variables it becomes independence of x_n. Thus the k_n, y_n integrals are in the nature of Fourier transformations, whereas the x_n integral is an imposition of gauge invariance.

One can also do the integral over k_0:

$$\int dk_0 S_{1,1}(k_0, x_n) e^{ik_0 Y} = S_{1,1}(Y, x_n).$$

Note S depends explicitly on x_n but also implicitly on x_n through Y because Y depends on x_n and all the derivatives of $X(z)$. Thus S is a non-local object in that it depends on all the derivatives of X. To put it another way, specifying S requires specifying a curve $X(z)$, because no two curves will produce the same value for S for all x_n.

Thus the dependence on the infinite number of x_n coordinates effectively makes S non-local in z and therefore $X(z)$. Of course the relevant scale here

is the string scale so at low energies one can neglect the higher derivatives and effectively S becomes an ordinary local field. It is also possible to redefine fields so that this non-locality disappears [15].

Note also that x_n being along gauge directions we can fix gauge and set them to some fixed value. So there is no increase in the number of physical degrees of freedom. This is a desirable feature.

8 Conclusions

We have described a general construction that gives gauge invariant equations of motion, the gauge transformation prescription (in terms of loop variables) being the same as in II. This method was outlined in III but many of the details had not been worked out. One of the problems that was left unsolved was whether the map from loop variables to space time fields is unambiguous. In particular it was not obvious that there was a map that correctly reproduced the gauge transformations. The results of the present paper indicate that it is indeed possible to define space time fields and their gauge transformations consistently. There are two crucial ingredients. One is that one has to carefully keep all the singular terms that are normally discarded by "normal ordering". We have to keep a finite cutoff in order for this procedure to make sense. This is not unexpected - we already know that in order to define off-shell Green functions in this approach, one needs a finite cutoff [19, 21, 22]. As shown in [13], even U(1) gauge invariance of the massless vector is violated when a finite cutoff is introduced in order to go off-shell, and one needs to introduce massive modes to restore gauge invariance. In the loop variable formalism all the modes are present from the start and there is no problem. Gauge invariance is present, on or off-shell. However the exact value of the Koba-Nielsen integral will depend on the cutoff prescription. Presumably these are equivalent to field redefinitions (of the space time fields).

The second ingredient is that the string-field Ψ and thus the space-time fields are functions of the gauge coordinates x_n. This is crucial for consistency of the definitions of gauge transformations. Thus effectively "space-time" has become infinite dimensional!

At this stage we have a non-trivial interacting theory with an infinite tower of higher spin gauge fields and a large gauge invariance. By construction these modes are essentially [6] those of the open bosonic string (including

[6] "essentially" because of a technicality that is discussed in I

the auxiliary ones). Nevertheless we have not proved that the amplitudes of this theory are those of the bosonic string. We have to demonstrate that the procedure of "dimensional reduction" that worked for the free case goes through here also i.e. without loss to gauge invariance. If this works out then we are guaranteed that the on-shell amplitudes are those governed by the bosonic string simply because the two dimensional correlators that are being calculated here are identical to those of the bosonic string amplitude calculation. There are arguments that this is in fact the case [15]. Furthermore as the gauge invariance does not use any on-shell conditions, these amplitudes are guaranteed to be gauge invariant off-shell also. Thus we have an off shell formulation. Further tests of the consistency of this will involve checking loop amplitudes. This is work for the future.

The main advantages are that the prescription for writing down the equations and gauge transformation laws are fairly straightforward. The gauge transformations written in terms of loop variables seem to have some geometric meaning - they look like local scale transformations. The interactions look as if they have the effect of converting a string to a membrane. The fields also appear massless in one higher dimension. These are intriguing features. [17, 16, 18] Finally, assuming the above issues are resolved satisfactorily, one has to see whether this formalism provides any insight into the various other issues that have become pressing in string theory, such as duality. Some of the structure observed in [14] may be relevant for this.

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A Appendix: Covariant Taylor Expansion

We derive the covariant Taylor expansion for $\tilde{\Sigma}(z(\sigma_1), z(\sigma_2), \sigma_1, \sigma_2)$. We first derive a Taylor expansion for $Y(z)$ and then use it to obtain a Taylor expansion for $\tilde{\Sigma}$.

A.1 Taylor Expansion for Y

Ordinary Taylor expansion gives,

$$Y(z+a) = Y(z) + a\frac{dY}{dz} + \frac{a^2}{2}\frac{d^2Y}{dz^2} + ... \tag{A.1}$$

$$Y \equiv \sum_{n\geq 0} \alpha_n \tilde{Y}_n \tag{A.2}$$

where $\tilde{Y}_n \equiv \frac{1}{(n-1)!}\frac{\partial^n X}{\partial z^n}$ and α_n satisfy [1]

$$\sum_{n\geq 0}\alpha_n t^{-n} = e^{\sum_{n\geq 0} x_n t^{-n}}$$

$$\frac{\partial \alpha_n}{\partial x_m} = \alpha_{n-m} \ ,$$

$$\frac{dY}{dz} = \tilde{Y}_1 + \sum_{n=1} n\alpha_n \tilde{Y}_{n+1} \ ,$$

$$= \tilde{Y}_1 + \sum_{n=1} n x_n \frac{\partial}{\partial x_{n+1}} Y \ . \tag{A.3}$$

In the above we have used $\sum_n [n x_n \frac{\partial}{\partial x_n}]\alpha_m = m\alpha_m$.

Differentiating (A.3) gives

$$\frac{d^2 Y}{dz^2} = \tilde{Y}_2 + \sum_{n=1} n x_n \frac{\partial}{\partial x_{n+1}}(\frac{dY}{dz})$$

Plugging in (A.3)

$$= \tilde{Y}_2 + \sum_{n=1} n x_n \frac{\partial}{\partial x_{n+1}}(\tilde{Y}_1 + \sum_{m=1} m x_m \frac{\partial Y}{\partial x_{m+1}})$$

$$\frac{d^2 Y}{dz^2} = \tilde{Y}_2 + \sum_{n,m=1} nm x_n x_m \frac{\partial Y}{\partial x_{n+m+2}} + \sum_{m=2} m(m-1) x_{m-1} \frac{\partial Y}{\partial x_{m+1}} \tag{A.4}$$

Adding (A.2),(A.3),(A.4) gives the first few terms of a Taylor series, except that we would like to express $\tilde{Y}_i$ in terms of Y_n in order to make the expression covariant.

$\tilde{Y}$ **in terms of Y:**

We first write

$$\alpha(t)\partial_z X(z+t) = \sum_{n,m\geq 0} t^{m-n}\alpha_n \tilde{Y}_{m+1} \tag{A.5}$$

Let $\beta(t) = \sum_{p\geq 0}\beta_p t^{-p}$. Let us evaluate $\int \frac{dt}{t}\beta(t)\alpha(t)\partial_z X(z+t)$.

$$\int \frac{dt}{t}\beta(t)\alpha(t)\partial_z X(z+t) = \sum_{n,p,m=n+p} \beta_p \alpha_n \tilde{Y}_{m+1}$$

$$= \sum_{m,p\geq 0} \beta_p \frac{\partial}{\partial x_{p+1}} \alpha_{m+1} \tilde{Y}_{m+1}$$

So

$$\int \frac{dt}{t} \beta(t)\alpha(t)\partial_z X(z+t) = \sum_{p\geq 0} \beta_p \frac{\partial}{\partial x_{p+1}} Y \tag{A.6}$$

Let us choose $\beta(t)$ having the property $\beta(t)\alpha(t) = t^{-s}; s \geq 0$ and call it $\beta^s(t)$ with the expansion

$$\beta^s(t) = \sum_{p\geq s} \beta^s_p t^{-p}$$

Then (A.6) will become

$$\tilde{Y}_{s+1} = \sum_{p\geq s} \beta^s_p Y_{p+1} \tag{A.7}$$

Thus if we determine β^s_p we obtain the required expansion.

To determine β^s_p we note that

$$\beta^s_p t^{-p} = \beta^s(t) = t^{-s}\alpha^{-1}(t)$$

$$= t^{-s} e^{-\sum_n x_n t^{-n}}$$

$$= \sum_{n\geq 0} \alpha_n(-x_n) t^{-n-s} = \sum_{p\geq s} \alpha_{p-s}(-x_n) t^{-p}$$

$$= \frac{\partial}{\partial(-x_s)} \alpha_p(-x_n, t)$$

This gives

$$\frac{\partial \alpha_p}{\partial x_s} \mid_{x_n \to -x_n} = \alpha_{p-s}(-x_n) = \beta^s_p \tag{A.8}$$

Thus for instance

$$\beta^0_0 = 1$$

$$\beta^0_1 = -x_1$$

$$\beta^0_2 = \frac{x_1^2}{2} - x_2$$

$$\beta^0_3 = -\frac{x_1^3}{6} + x_2 x_1 - x_3$$

Therefore

$$\tilde{Y}_1 = Y_1 - x_1 Y_2 + (\frac{x_1^2}{2} - x_2) Y_3 + ... \tag{A.9}$$

It is easy to see that

$$\beta_p^s = \beta_{p-s}^0$$

and so all the coefficients are easily determined.

Similarly

$$\frac{\partial}{\partial x_n}\beta_r^s = -\beta_{r-n}^s$$

Using this it is easy to see that

$$\frac{\partial}{\partial x_n}\tilde{Y}_s = 0$$

as it should be.

Using the above results one obtains

$$\begin{aligned} \frac{dY}{dz} &= \tilde{Y}_1 + \sum_{n=1} n x_n \frac{\partial}{\partial x_n} Y_1 \\ &= \sum_{r\geq 0} (\beta_r^0 + r x_r) Y_{r+1} \\ &= \sum_{r \geq 0} \gamma_r^0 Y_{r+1} \end{aligned} \tag{A.10}$$

Using the fact that $\frac{\partial^2}{\partial x_n \partial x_m} Y = \frac{\partial}{\partial x_{n+m}} Y$ one can easily verify that $\frac{\partial^2}{\partial x_n \partial x_m}\frac{dY}{dz} = \frac{\partial}{\partial x_{n+m}}\frac{dY}{dz}$. This is as it should be because the operations of differentiating w.r.t. z and w.r.t x_n commute.

A.2 Taylor Expansion of $\tilde{\Sigma}$

We now use this to obtain the covariant expansion of $\tilde{\Sigma}$. $\tilde{\Sigma}$ was defined in section III

$$\tilde{\Sigma}(z_1, z_2) = \oint du \omega(u) < \partial_u X(u) Y(v-a) >< \partial_u X(u) Y'(v+a) > \tag{A.11}$$

where $z_2 - z_1 = 2a$ and $z_1 + z_2 = 2v$ and the contour encircles both points. We will call the gauge coordinates x_n at z_1 and y_n at z_2. The prime on Y indicates indicates that it is a function of y_n.

We will use the shorthand notation $< Y(v-a)Y'(v+a) >$ for the above definition of $\tilde{\Sigma}$. Thus we have the following Taylor expansion for $\tilde{\Sigma}$:

$$< Y(v-a)Y'(v+a) >=< (Y(v) - a\frac{dY}{dv} + \frac{a^2}{2}\frac{d^2Y}{dv^2} + ...)$$

$$(Y'(v) - a\frac{dY'}{dv} + \frac{a^2}{2}\frac{d^2Y'}{dv^2} + ...) >$$

Plugging in the Taylor expansions for Y that has been derived in this Appendix we get

$$= \bar{\Sigma}(v) + a\sum_{r\geq 0}(\gamma_r^{0'}\frac{\partial}{\partial y_{r+1}} - \gamma_r^0\frac{\partial}{\partial x_{r+1}})\bar{\Sigma}(v) +$$

$$a^2[-\sum_{r,s\geq 0}\gamma_r^0\gamma_s^{0'}\frac{\partial^2}{\partial x_{r+1}\partial y_{s+1}}\bar{\Sigma} + \frac{1}{2}\sum_{r\geq 0}(\gamma_r^1\frac{\partial}{\partial x_{r+1}} + \gamma_r^{'1}\frac{\partial}{\partial y_{r+1}})\bar{\Sigma}] + O(a^3) \tag{A.12}$$

Here as before $\bar{\Sigma}(z, x_n, y_n) \equiv \tilde{\Sigma}(z, x_n, z, y_n)$

This is what has been used in section VI.